ELEMENTS

OF

GEOMETRY AND MENSURATION,

WITH

EASY EXERCISES,

Designed for Schools and Adult Classes.

IN THREE PARTS.

PART I.—GEOMETRY AS A SCIENCE.
PART II.—GEOMETRY AS AN ART.
PART III.—GEOMETRY COMBINED WITH ARITHMETIC.
(MENSURATION).

BY

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ADVERTISEMENT TO PART I.

The following short Treatise on *Geometry as a Science* makes no pretence of entering into competition with *Euclid's Elements*—the most wonderful book perhaps, with one exception, in existence. But as it cannot be denied, that *Euclid* presents Geometry in a diffuse and somewhat repulsive form, whereby a large proportion of those, who ought to be acquainted with the subject, are deterred from venturing upon it at all, I have thought that good service might be rendered to the cause of popular education by framing a work, which shall neither terrify by its size, nor repel, as Euclid does, by a studied avoidance of all practical illustration. At the same time I have endeavoured, except in a single instance, to preserve the *strictness* of the ancient geometers, at least to the extent of laying down a solid and trustworthy *foundation* for that which is to follow. I cannot discover any good reason, why the *mensuration* taught in our Schools should be built, as it mostly is, upon no foundation but the *memory* only; I think it need not, and I am sure it ought not, to be so. But as it is, we reap the fruits of this bad system of mental culture in the very general ignorance of *right principles* of construction and design, which notoriously prevails among *English* artists and workmen. Public attention has been lately directed to the necessity of removing this stigma from our character as a people by the institution of

*Schools of Design and Practical Art**. Let me urge upon the managers of such Schools the expediency of beginning their work *at the right end.* Let *principles* be taught before *rules.* Let *Geometry as an Art* be systematically *preceded* by *Geometry as a Science.* Then, but not till then, we may hope to see the desired result in the improved taste and skill of our designers, and to be saved the continuance of that sense of humiliation which every Englishman must experience on reading the statement here subjoined.

T. L.

MORTON RECTORY, ALFRETON,
April 20, 1854.

* On a late public occasion, at the inauguration of one of these Schools, the Duke of Argyll remarked that "a very large proportion of the works of art preparing for the Crystal Palace are being executed almost entirely by *foreign* artists, and that our manufacturers also have been obliged to send *abroad* for designs; and, as he was convinced that there was no natural disqualification in our population for such work, he trusted that the defect would be remedied by the adoption of a more complete system of education."

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It is clear also that, having to treat of bodies, or parts of bodies, *in respect of magnitude* and *position*, we have to provide for taking *measurements* of various kinds; and hence is required a sort of geometrical language in the first onset, which must be learnt from the following Definitions:—

3. We measure a *distance* by a '*line*'; so that a *line* will represent any one of the dimensions *length, breadth, height, girth, depth*, or *thickness.* We do not inquire as to the *thickness* of the *line*, when used *for this purpose of measurement.* Hence the common

DEFINITION. *A* LINE *is length without breadth or thickness.*

It is not meant that any *line* we can actually *use* or *make* is *without breadth* or *thickness;* but that for *Geometrical* purposes, that is, as a measure of *length*, the *length only* of a *line* is *considered.*

Thus, for illustration, if the *length* of a room be in question, we regard not the fact of its being measured by a *broad* tape or a *narrow* tape—even the finest thread we can use will serve our purpose, if it be inextensible, —we expect the same result in each case, because it is *length only* we are concerned with. In the case here supposed, the broad tape is not inferior to the finest thread; but, as there are numberless other cases in which this is not so, (as will appear hereafter), the Definition of a '*line*' above given is the only one which can insure general accuracy of measurement.

4. Another term in common use in Geometry is '*point*', by which is meant generally no more than a place to start from, or to stop at, in drawing or measuring a *line*. A *point* hath *position* only, and is nothing for us to *measure;* and hence the common

DEFINITION. *A* POINT *hath no parts and no magnitude.*

It is true we cannot *exhibit* such a *point*, (because that which hath no magnitude cannot be visible to the human eye); but the more nearly the *points* we use in practice approach the strictness of this Definition, the more accurate, it is obvious, will be the measurements which begin or end at those points.

It follows, that each extremity of a *line* is a *point*.

5. *Lines* are of two kinds, *straight* and *crooked*.

A *straight line*, or, as it is often called, a *right line*, is the *direct*, that is, the *shortest*, line connecting the two extremities, or extreme *points*, of it.

A *crooked line* is not the direct line joining the two points which are its extremities. It may consist of two or more straight lines joined together thus,  or in some other way. Or it may be what is called a *curved* line, *no part being straight*, such as , or such as may be represented by a fine thread drawn tight round the trunk of a tree to measure its girth.

In speaking of *points* we distinguish one from another by using the letters of the alphabet to mark their position; and so also with regard to *lines* to mark either their position or extent, or both.

A *single* letter will fix or express a *point*, but *two* are mostly used to express a *straight line*. Thus, if we put *A* at one end of a *straight line*, *B* at the other end, the *points*, which are the extremities of that line, would be simply called the points *A* and *B*; and the *line* would be called the line *AB*.

Sometimes, however, a single letter may be used to denote a *line*, but not often.

6. SUPERFICIES, SURFACE, or AREA. These words all express the same thing, which is a subject for measurement; as, for instance, the *acre-age* of a field. It is obvious that this will depend upon the *length* and *breadth* of the field, but not at all upon the *depth* of the soil, or the *thickness* of the sod. And so we have the

DEFINITION. A SUPERFICIES, SURFACE, *or* AREA, *is that which hath only length and breadth.*

It is not meant that the *body* whose *superficies, surface*, or *area*, we are considering has only length and breadth, but that the dimensions of a *superficies, surface*, or *area*, are entirely dependent upon length and breadth, to the exclusion of *thickness, height*, or *depth*. Thus in speaking of the *quantity* of carpet which will cover a floor, the *thickness* of the carpet never enters into our consideration, but only the length and breadth.

Hence the expression '*superficial measure*' is always

understood to exclude *thickness.* Thus, for instance, the *area* or *surface* of this page, that is, the space upon it capable of receiving the impression of type, is manifestly independent of the *thickness* of the paper.

7. SURFACES are of two kinds, *plane* and *curved.*

A *plane surface* is one on which a *straight line* may be drawn *in any part* of it, *wholly coincident* with the surface. Or, in other words, if *any* two points are taken in the surface, and a *straight line* be drawn joining the two points, that line shall be *wholly* in the surface.

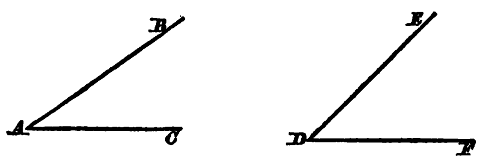
A *curved surface* is one, on which if points be taken and joined by lines lying wholly on the surface, those lines are found to be *curved lines.*

Thus the top of a table is a '*plane* surface'; but the boundary of a globe is a '*curved* surface'.

Observe, it is not necessary to a *curved* surface that *all* lines drawn on it should be *curved* lines; there may be *straight* lines in particular cases. For example, the surface of a round pillar is *curved*, but yet the lines drawn on it in the particular direction of the *length* of the pillar will be *straight* lines, whilst all others will be *curved.*

8. ANGLES. *A plane rectilineal angle* is formed by two straight lines, which *meet together,* but are not in the same straight line. The *angle* is the *measure* of the *inclination* of the one line to the other; but how that measure is *taken* does not concern us at present to know. All that is here required is to know how to *compare one angle with another, viz.:*

(1) That the *angle* formed by, or between, the lines *AB*, and *AC*, which meet at the point *A*, is *equal* to the *angle* between the lines *DE*, and *DF*, which meet at the point *D*, if, when the point *A* is 'applied to', or placed

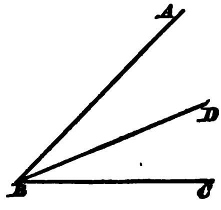


upon, the point *D*, and the line *AC* upon the line *DF*, then also the line *AB* coincides with *DE*.

(2) That the *angle* between AB and AC is *greater* or *less than* the *angle* between DE and DF, according as, when AC is applied to DF as before, AB falls farther from, or nearer to, DF, than DE does.

9. An *angle* is generally denoted, or expressed, by *three letters* of the alphabet, in the following manner: The middle letter invariably marks the point where the lines which form the angle meet together, and of the other two letters one is upon one of the lines and the other upon the other line.

Thus, if the lines BA, BC, BD, meet together at the same point B, the angle between BA, and BD, is called the angle ABD, or DBA, whichever we please, only taking care that B is the middle letter; the angle between BA and BC is called the angle ABC, or CBA; and the angle between BD and BC is called the angle DBC or CBD.



Sometimes, however, when only two lines meet together, forming only *one* angle, so that no mistake can arise as to the angle meant, that angle may be described by a *single* letter placed at the point where the lines meet. Thus, the angle formed by two lines which meet at the point A would be called '*the angle at A*'.

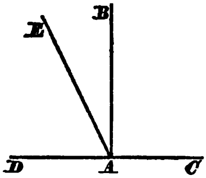
The *point* where the lines which form an angle meet together is called the *angular point*, or *vertex* of the angle; and ought to be carefully distinguished from the *angle itself*.

Observe, the *magnitude of an angle* does not at all depend upon the *length* of the lines by which it is formed, but only upon their *position*. Yet the lines must be some length to be lines at all.

10. If one of the lines which form an angle be extended in the same straight line from the angular point, so as to form a second angle on the same side of it adjacent to the former, and these angles are found to be *equal* (8)* to each other, then each of the angles is called

* This will be the mode of referring to a previous paragraph, or article, as it is usually called. In this case, it is meant that the reader look back to the paragraph numbered 8, and see that a method has been there explained of *comparing one angle with another*.

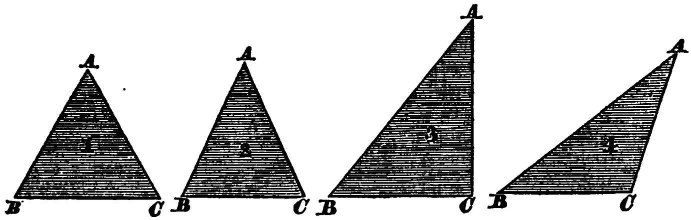
a right angle. Thus, if CA, one of the lines which form the angle BAC, be extended to a point D beyond A *in the same straight line,* and then the angle BAD is found to be equal to the angle BAC, each of these angles is *a right angle.* In this case also the line BA is called a *perpendicular to* the line CD; and again, AB is said to be *at right angles with CD.*



An *obtuse angle* means an angle *greater than* a right angle, as EAC. (8).

An *acute angle* means an angle *less than* a right angle, as EAD. (8).

11. TRIANGLES. A *plane surface* bounded by three straight lines meeting together at their extremities, so as entirely to enclose a space, is called a *triangle;* and the three straight lines are called the *sides* of the triangle. Thus each of the following figures is called the *triangle ABC,* whose *sides* are AB, AC, BC, the letters A, B, C, being at the three *angular points.*



When the *three* sides are equal to each other, the triangle is called *equilateral,* or equal-sided, as in fig. 1, where $AB = AC = BC$*.

When *two* sides only are equal, as in fig. 2, where $AB = AC$, and BC is unequal, the triangle is called '*isosceles*', which signifies '*equal-legged*', as if the triangle

* The following abbreviations will be used throughout the book :—
= for '*equals*', or '*is equal to*'.
+ for '*added to*', or *to be added*'.
$\angle$ for '*angle*'.
$\therefore$ for '*therefore*'.

were supposed to *stand* upon BC, as a *base*, with two legs AB, AC.

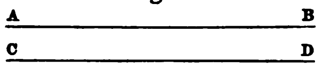
A *triangle*, as the name implies, has also three *angles* within it, as the '*angle at A*', the '*angle at B*', and the '*angle at C*', or $\angle BAC$, $\angle ABC$, and $\angle BCA$: and triangles have received other distinctive names, besides those mentioned above, after the names of one or more of these *angles*. Thus,

A triangle, which has one* of its angles a *right angle*, is called a *right-angled triangle*, as ABC fig. 3, where the 'angle at C' is a *right* angle.

A triangle, which has one of its angles an *obtuse angle*, is called an *obtuse-angled triangle*, as ABC fig. 4, where the 'angle at C' is an *obtuse* angle.

A triangle, which has each of its angles *acute*, is called an *acute-angled triangle*, as ABC figs. 1 and 2.

12. PARALLEL *straight lines* are such as, being in the same plane, never meet though produced ever so far both ways. Thus the straight lines

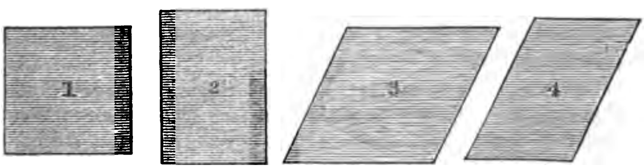


AB, CD, are *parallel* to each other, if, being both on the plane of this paper, they never meet however far produced either towards the right hand or the left.

13. PARALLELOGRAMS. A *parallelogram* is a plane surface bounded by *four* straight lines, called its *sides*, of which each opposite two are *parallel*.

There are several kinds of parallelograms:—viz.

(1) A *square* is a *parallelogram*, which has all its sides equal and all its angles *right* angles, as fig. 1.



(2) An *oblong*, or *rectangle*, is a *parallelogram*, which

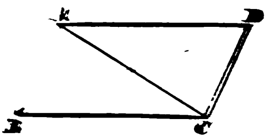
* It will be seen hereafter that no triangle can have more than one *right* angle.

[illegible] its angles right angles, but not all its sides, only its opposite sides equal to each other, as fig. 2.

[illegible], or square, is a parallelogram, which has all its sides equal, and each of its angles is a right angle, as fig. 3.

[illegible] rhomboid, is a parallelogram, which has its opposite sides, but not all its sides, equal, and none of its angles is a right angle, as fig. 4.

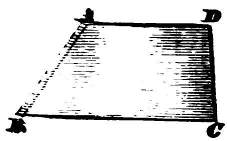
A parallelogram is generally denoted or expressed by any two of the four letters at [illegible] which are placed at the four corner points. Thus the parallelogram here traced would be called the parallelogram ABCD, or ADCB, or BADC, &c. whichever we please.



14. A diagonal, or diameter, of a parallelogram is the straight line joining two of its opposite angular points. Thus *AC*, and *BD* are the diagonals, or diameters, of the parallelogram *ABCD*, in the preceding fig.

Also the side *BC*, upon which the parallelogram may be supposed to stand, is sometimes called its *base*.

15. A plane surface bounded by four straight lines of which two only are parallel, is called a trapezium, as *ABCD*, where *AD* is parallel to *BC*, but *AB* is not parallel to *CD*.

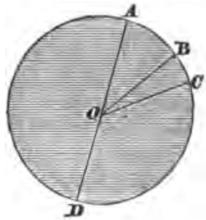


16. CIRCLES. A *circle* is a *plane surface* bounded by a *curved line*, such that every point in this line is equally distant from a certain point within the figure [illegible] the centre of the circle.

[illegible]ved boundary is called the *circumference* of [illegible] and the straight line which measures the [illegible] the *centre* to the *circumference* is called [illegible] the circle. Any straight line drawn through [illegible]d terminated both ways by the *circumference* [illegible]*ameter* of the circle.

[illegible] fig. here traced, the area or surface in the

plane of the paper bounded by the curved line *ABCDA* is a *circle*, when from the *centre O* all straight lines to the *circumference*, as *OA*, *OB*, *OC*, *OD*, are equal to each other.



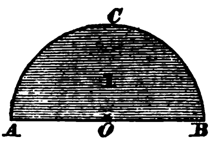
Any one of the lines *OA*, *OB*, *OC*, *OD* is the *radius*, and any radius, as *AO*, extended in the same straight line to meet the circumference in *D*, that is *AD*, is a *diameter*, of the circle.

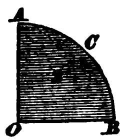
17. Hence it is plain, that a *circle* may be traced by means of a string, one end of which is kept fixed in a certain point as the *centre*, while the other is made to revolve and trace out the *circumference*, the string being kept perfectly tight. The same thing is also done by the ordinary *compasses*.

18. A *semi-circle* is the half of a circle, bounded by the half of the circumference and the diameter joining its extremities.

A *quadrant* is the quarter of a circle, or the half of a semi-circle, bounded by the *fourth*-part of the circumference and two radii joining its extremities with the centre.

Thus fig. 1 is a *semicircle*, and fig. 2 is a *quadrant*,





where *O* is the *centre* of the circle in each case; and whilst *ACB* is half of the *whole* circumference in the former, it is a quarter of it in the latter.

An *arc* of a circle is a *portion* of the *circumference*.

It may be observed here, that although *two* letters are sufficient to express a *straight* line, *three* or more are generally required for a *curved* line; and for an obvious reason, because between any two points there is only *one straight* line, but an infinite number of *crooked* lines, so that the extreme points entirely *determine* the former but not the latter.

19. It will be found, hereafter, that we often, for shortness, call the *circumference* of a circle the *circle*, which, though convenient, is not a *correct* way of speaking. In the same manner it is not unusual to hear persons speak of a *triangle*, *square*, or other *plane surface*, when, in fact, they mean no more than the *boundary* of the figure in each case.

Let it, then, be borne in mind, that in strictness a *circle* does not *consist* of one *curved line* merely, called the *circumference*, but that it is the whole inner area *bounded by* that line.

So, again, a *triangle* does not *consist of* three straight lines called *sides*, but is the *whole inner area bounded by* those sides. And similarly with respect to other plane *surfaces*.

20. EUCLID'S '*Postulates*' must now be admitted as truths *to be granted* without proof, viz.

I. A straight line may be drawn on a given plane surface from any one point to any other point.

II. A terminated straight line may be 'produced', that is, extended, to any length in a straight line.

III. A circle may be 'described' with any centre, and any given length, or line, for its radius.

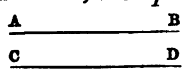
Granted that we can do these three things, and we will *assume* nothing further in the construction and treatment of Geometrical figures.

In 'describing' a circle, by the third *Postulate*, we *trace* out the *circumference* which is the *boundary* of the circle. Of course we can trace a *part*, as well as the whole, that is, any *arc* of the circle.

21. EQUALITY of LINES, AREAS, and ANGLES.

It is *evident* that magnitudes which coincide in every part are equal to one another. This is a received *axiom* which admits of no dispute. It is the simplest notion we have of equality.

Hence the two *straight lines AB*, and *CD*, are *equal* to one another, if, when *CD* is placed upon *AB*, so that the point *C* is upon *A* and *CD* upon *AB*, the point *D* is found to coincide with the point *B*.

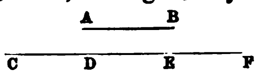


In like manner two *areas* are equal to one another, when they can be made to coincide in every part, that is, when one can be made exactly to cover the other, and no more. For example, all the *pages* of this book are exactly equal to one another. But areas may be *equal* also, when they are not exactly alike, (as the pages of the same book are,) but can be *made so* by a *different arrangement* of the *parts* of one or both. For it is evident, that this page might be cut up into many parts, without at all altering the total area; and those parts might be arranged so as to form a great variety of plane figures having precisely the same area, but with a different boundary. Thus, if we have a square and a triangle, and we can cut up the square, so as with the parts exactly to cover the triangle, the area of the square is *equal* to that of the triangle. Or, again, two *triangles*, which to the eye appear unequal, may yet be equal, and shall be so, if by a different arrangement of parts they *can be made to coincide.*

The Equality of *Angles* has been already defined in (8).

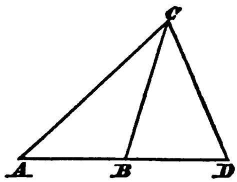
22. Addition, Subtraction, &c. of Lines, Areas, and Angles.

It follows from (21) that *lines, areas,* and *angles,* may be *added* together, *subtracted* from each other, *multiplied,* or *divided,* like other magnitudes. Thus if AB, CD be two straight lines *equal* to one another, 'produce' CD indefinitely towards D, then by applying AB to it so that A is upon D, and AB upon DE, we find DE *equal* to AB, and $\therefore$ CE is plainly *twice* AB. Again, if $EF = AB$, then $CF =$ *three* times AB, and so on. And thus we *multiply* the line AB. Obviously also CD is *one-third* of the line CF; that is, a line may be *divided.* Again, that *lines* may be *added* or *subtracted* is plain enough, for $AB + CD = CD + DE = CE$; and AB taken from $CE = CD$.



The same principle, viz. that 'magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another', leads to the conclusion that, in like manner *areas* and *angles* may be *added, subtracted, multiplied,* or *divided.*

Thus, for instance, if AB, BD be in the same straight line, so that ABC, BCD, ACD are three distinct *triangles*, it is plain that the two areas ABC, BCD, exactly cover the same space as the area ACD, and $\therefore$ the two areas ABC, BCD may be *added* together, and their *sum* will be the area, or triangle, ACD.



Similarly, if the area BCD be *subtracted* from the area ACD, the *difference* is the area ABC.

Again, if area ABC = area BCD, then area ACD is *double* of the area ABC; and area ABC = *half* of area ACD.

Angles likewise are magnitudes which may be *added*, *subtracted*, &c. Thus, $\angle ACB + \angle BCD = \angle ACD$. And $\angle BCD$ taken from $\angle ACD$ leaves $\angle ACB$. Also if $\angle ACB = \angle BCD$, then $\angle ACD$ is *double* of $\angle ACB$.

QUESTIONS ON THE PRECEDING DEFINITIONS, &c.

(1) What does *Geometry* treat of? To what properties of bodies is it restricted?

(2) Define a '*line*'; can it be *exhibited* in practice? If not, why not?

(3) How many different kinds of *lines* are there? Give an example of each.

(4) Define a '*point*'. Can it be exhibited to the eye? If not, why not? Give an example of a '*point*'.

(5) Define '*superficies*', '*surface*', or '*area*'. Give an example.

(6) How many kinds of '*surfaces*' are there? Give an example of each. By what general rule are they distinguished from each other?

(7) What is meant by '*the line AB*'? Is it the same as *the line BA*?

(8) What is an '*Angle*'? Is it a magnitude admitting of increase or decrease? Exhibit *two* angles; and say which is the greater, and why.

(9) What is meant by '*the angle ABC*'? Is it the same as 'the angle *CBA*'? Is it the same as 'the angle *ACB*'?

(10) Define a *right angle*, and exhibit it. Can one right angle be *greater* than another right angle? What is the way of determining whether one angle is *greater* than another?

(11) What are the *names* by which certain *angles* are distinguished? Exhibit an angle of each sort.

(12) Explain clearly the difference between the *angle* *ABC* and the *triangle* *ABC*.

(13) By what names are *triangles* distinguished according to their form? Exhibit a triangle of each sort.

(14) Does the magnitude of an *angle* depend upon the magnitude of the *lines* by which it is formed?

(15) How many *lines* are necessary to form an *angle*? How many to form a *triangle*?

(16) How many *angles* are there in a triangle? Does the magnitude of a *triangle* depend upon the magnitude of the *lines* which form its three *angles*?

(17) Does the word *triangle* mean '*three angles*' in such a sense as to signify that the triangle is *made up* of the three angles, *so as to be equal to them*?

(18) Define *parallel* straight lines; and give an example.

If a straight line were drawn on the ceiling, and another on the floor, these two lines being produced ever so far both ways would never meet. Would they necessarily be *parallel*? Does the definition exclude such?

(19) How many kinds of *parallelograms* are there? What is the distinctive character of all, and of each? Exhibit each separately, and fully describe it.

(20) How many *letters* are used to *denote* a particular parallelogram, and where are they placed? Give an example.

(21) What is meant by the '*base*' of a parallelogram?

(22) Define a '*circle*'; and explain clearly the difference between a *circle* and the *circumference* of a circle.

(23) How many letters are required to denote an *arc* of a circle? Why will not *two* serve, as in the case of a *straight* line? Where are the letters placed?

(24) What is the object of Euclid's three *Postulates?*

(25) Upon what *axiom* does the *Equality* of geometrical magnitudes depend?

(26) Can *one angle* be *equal* to *two* other *angles*, or to *three?* Explain clearly.

(27) Is it *possible* for a *triangle* to be *equal* to a *square?* If so, say how.

(28) How many *angles* are there in a *parallelogram?* Is the parallelogram *equal* to the *sum* of its *angles?*

(29) Is a *semi-circle* a *line* or an *area?*

(30) Is an *angle* an *area?* If so, how do you understand the statement at the end of (9) page 5?

(31) Can one triangle be *added* to another? If *triangles* be added together will the resulting *sum* necessarily be a *triangle?*

(32) Is a triangle *equal* to the sum of its three sides?

(33) What is the difference between an *angle*, and a *corner?* What is the *geometrical* name for the latter?

EXPLANATION OF TECHNICAL TERMS USED IN GEOMETRY.

(1) To '*describe*' a certain geometrical figure, means to *construct*, or *trace*, it on a plane surface, as a board or sheet of paper.

(2) A '*given*' line means a line 'given' sometimes in *position*, sometimes in *magnitude*, sometimes in *both*, according to circumstances; and the word '*given*' means *fixed* or *known*.

(3) A '*proposition*' is something *proposed* to be done; so that the heading of each separate article in the following section may be called a *proposition*. Sometimes '*propositions*' are distinguished into two kinds; they are called *problems*, when something is proposed to be *constructed* or *made;* and they are called *theorems*, when some proposed statement is required to be *proved*.

(4) '*Corollary*' signifies an *after-conclusion* beyond what is due, following obviously without any or much further proof from what has been already done or proved.

(5) *à fortiori* means '*by so much the more*'. Thus, if *A, B, C* represent three geometrical magnitudes, and we know that *A* is greater than *B*, having *proved* that *B* is greater than *C*, we conclude, *à fortiori*, that *A* is greater than *C*.

(6) The '*converse*' of a proposition is when the *conclusion* is turned into an *assumption*, and the previous *assumption* is made the *conclusion*. Thus to the proposition "The angles at the base of an isosceles triangle are equal to one another" the *converse* would be "Shew that, if the angles at the base of a triangle are equal to one another, the triangle is isosceles".

(7) '*reductio ad absurdum*', (*reducing to an absurdity*), is a particular mode of demonstration often used by Euclid. It may be briefly explained by the following case:—Required to shew that two geometrical magnitudes, represented by *A* and *B*, are equal to one another. We argue thus. If *A* is *not* equal to *B*, then *A* and *B* must be unequal. *Suppose* them unequal, and proceeding upon this assumption we arrive, by means of acknowledged axioms and legitimate reasoning, at an *absurd* conclusion, such as, for instance, that a *portion* of a magnitude is greater than the *whole*. If then the supposition that *A* and *B* are *unequal* legitimately leads to such a conclusion, it is plain that that supposition cannot stand; and therefore the only alternative is that $A = B$.

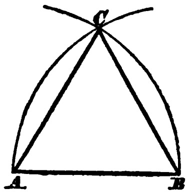
(8) To '*produce*' a given straight line is to continue or extend it, so that the part added may be in one and the same straight line with the given line. Thus a *radius* of a circle, continued through the centre to meet the circumference again, until it becomes a *diameter*, is said to be *produced*.

(9) An '*axiom*' is a statement of an admitted truth, so plain and unquestionable as to need no demonstration, as long as words mean what they do; as that, for instance, "the whole of any magnitude is greater than a part of the same magnitude"—or, again, that 'two is greater than one'. Such truths do not specially belong to *Geometry*, but are practically interwoven with almost every operation of daily life.

STRAIGHT LINES AND RECTILINEAL PLANE FIGURES.*

23. PROPOSITION I. *To describe an equilateral triangle upon a given straight line*†.

Let AB be the given straight line, which is to be one side of the triangle; with centre A and radius AB (POST. III. 20) trace a portion of the circumference of a circle on that side of AB on which the triangle is required; with the same radius and with centre B trace another portion of the circumference of a circle on the same side of AB,



and intersecting the former in the point C; join the points A and C by the straight line AC (POST. I.), and B and C by the straight line BC; then ABC shall be the equilateral triangle required.

For since B and C are points in the circumference of the same circle whose centre is A, $AB = AC$, (Def. 16); again, since A and C are points in the circumference of the same circle whose centre is B, AB or $BA = BC$;

$$\therefore\ AC = AB = BC,$$

or the three sides of the triangle ABC are equal to each other, that is, ABC is an *equilateral triangle* and it is described upon the straight line AB.

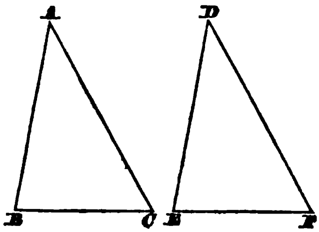
24. PROP. II. *If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles formed by those sides equal to one another, they shall also have their bases, or third sides, equal; and the two triangles shall be equal; and their other angles shall be equal, each to each, viz. those to which the equal sides are opposite.*

Let ABC, DEF be two triangles, in which the side

* A *rectilineal* plane figure means a *plane surface* (7) *bounded by straight lines*. According to the *number* of such lines, forming its *boundary*, each figure receives its distinctive name.

† The Author does not deem it advisable to deviate much from *Euclid's* mode of expression, but rather to explain it, when it appears necessary, in a note. Thus, in this instance, to '*describe*' a triangle means to *construct* or *trace* it; and '*upon a given straight line*' means *so as to have that straight line for its* BASE. Also '*a given* straight line' means here *a line fixed both in position and magnitude.*

$AB=$ the side DE, the side $AC=$ the side DF, and $\angle BAC = \angle EDF$. Suppose the triangle ABC to be laid upon the triangle DEF, in such manner that the point A is upon the point D, and the line AB upon DE; then the point B will fall

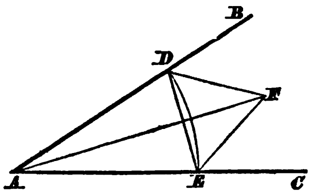


upon E, because $AB=DE$. Again, since AB falls upon DE, AC will also fall upon DF, because $\angle BAC=\angle EDF$, (8.) Since, then, the point A is upon the point D, and the line AC upon DF, the point C shall fall upon F, because $AC=DF$. Hence, since B is upon E, and C upon F, the line BC must coincide with EF, because BC and EF are *straight lines* between the same, or coincident, points. Therefore the *triangles* coincide, and are *equal, in all respects*, as stated above.

Cor. Hence, also, if two triangles have the *three* sides of the one equal to the *three* sides of the other, each to each, in the same order, the two triangles will be equal, and their angles likewise will be equal, each to each, viz. those to which the equal sides are opposite. For it is evident from what has been shewn above, that such triangles, applied to each other as in the former case, will coincide in every part, and therefore be equal in all respects*.

25. Prop. III. *To bisect a given angle*†, *that is, to divide it into two equal angles.*

Let BAC be the given angle; it is required to bisect it. In AB take any point D, and with centre A and radius AD describe an arc of a circle cutting AC in the point E; join the points D, and E, by the straight line DE, and upon DE describe the equilate-



* Euclid does not seem to have considered this sufficiently evident, and therefore proves it by the process, usually called *reductio ad absurdum*, before explained.

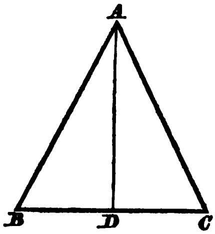
† *Given*, that is, by being traced on a given plane surface.

ral triangle DEF (23); then join AF, and the angle BAC is bisected by the line AF.

For, since D and E are points in the circumference of the same circle whose centre is A, $AD = AE$; and since DEF is an equilateral triangle, $DF = EF$. Therefore in the two triangles ADF, AEF, the three sides AD, DF, AF are equal to the three sides AE, EF, AF, each to each, in order; so that (24, Cor.) the two triangles are equal in all respects, and the angle DAF between AD, AF, is equal to the angle EAF between AE, AF. Therefore the angle BAC is divided into two equal angles by the straight line AF.

23. Prop. IV. *The angles at the base* of an isosceles triangle are equal to one another.*

Let ABC be an isosceles triangle, in which the side $AB =$ the side AC, and BC is the third side, or *base*; the angle ABC shall be equal to the angle ACB.



Bisect the angle BAC by the straight line AD (25), D being the point where AD meets the base BC. Then, since $AB = AC$, and $\angle BAD = \angle CAD$, we have two triangles ABD, ACD, in which the two sides BA, AD are equal to the two sides CA, AD, each to each, and the angle formed by the two sides of the one equal to the angle formed by the two sides of the other, $\therefore$ the triangles are equal in all respects (24), and the angles are equal, each to each, to which the equal sides are opposite; and

$$\therefore \angle ABD = \angle ACD,$$

or, which is the same thing, $\angle ABC = \angle ACB$.

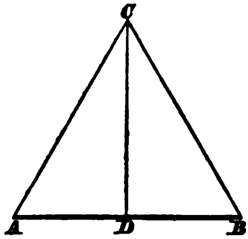
Cor. Hence every *equilateral* triangle is also *equiangular*; and, conversely, every *equiangular* triangle is also *equilateral*.

27. Prop. V. *To bisect a given finite† straight line, that is, to divide it into two equal straight lines.*

* The *base* in an *isosceles* triangle is restricted to one side, viz. the *unequal* side, on which the two equal sides may be supposed to stand, except when the triangle is *also equilateral*, in which case *any* side may be taken as the base.

† '*A given finite straight line*' means a straight line fixed *both* in *position* and *magnitude*.

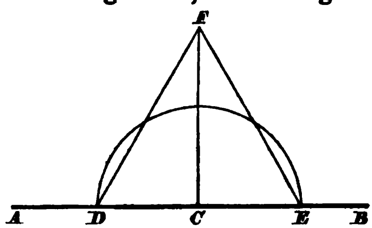
Let AB be the given straight line. Upon AB describe the equilateral triangle ABC (23); bisect the angle ACB by the straight line CD meeting AB in D (25); then AB is bisected in the point D.



For AC, CD are equal to BC, CD, each to each, and $\angle ACD = \angle BCD$, $\therefore$ the triangles ACD, BCD are equal in all respects (24); and $\therefore$ $AD = BD$, being the sides opposite to the equal angles ACD, BCD; that is, AB is divided into two equal parts in the point D.

28. PROP. VI. *To draw a straight line at right angles to a given straight line* from a given point in it.*

Let AB be the given straight line, and C a given point in it. It is required to draw a straight line from C *at right angles* to AB. In AC take any point D, and with centre C, and radius CD, describe an arc of a circle cutting the line AB in D and E; upon DE describe the equilateral triangle DEF (23); and join FC; CF shall be at right angles to AB.



For, since $CD = CE$ (16), and $DF = EF$ (23), by construction, in the two triangles DCF, ECF, DC, CF are equal to EC, CF, each to each, and the third side DF is equal to EF, $\therefore$ the triangles are equal in all respects, (24, Cor.) and $\angle DCF = \angle ECF$, to which the equal sides DF, EF are respectively opposite, and $\therefore$ each of them is a *right angle* (10), that is, CF is at right angles to AB.

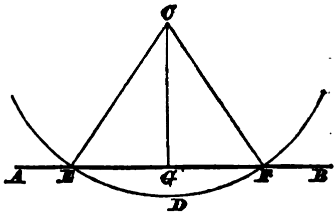
29. PROP. VII. *To draw a straight line* † *perpendi-*

* This straight line is required to be *given* in *position* only.

† Whether a certain straight line is drawn *at right angles*, or *perpendicular*, to another straight line, depends upon the simple fact, whether it be drawn, from a known point in the latter line itself, *away from the line*; or from a known point without it *towards the line*.

cular to a given straight line of unlimited length from a given point without it.

Let AB be the given straight line, and C a given point without it, from which it is required to draw a *perpendicular* to AB. Take a point D on the other side of AB, and with centre C and radius CD, describe a circle cutting AB, or AB produced, in E and F; join CE, CF, and bisect $\angle ECF$ by the line CG, meeting AB in G. Then CG shall be *perpendicular* to AB.



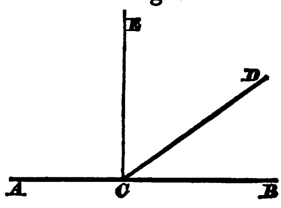
For $CE = CF$ (16), and $\angle ECG = \angle FCG$, by construction, $\therefore$ in the triangles ECG, FCG, EC, CG are equal to FC, CG, each to each, and $\angle ECG = \angle FCG$, $\therefore$ the triangles are equal in all respects (24), and the angles equal to which equal sides are opposite, viz.

$$\angle EGC = \angle FGC,$$

and $\therefore$ each of them is a right angle, or CG is *perpendicular* to AB (10).

30. Prop. VIII. *The two angles which one straight line makes with another upon the one side of it are always equal to two right angles.*

Let the straight line CD meet the straight line AB in the point C, and make with AB on the one side of it the angles ACD, BCD; these are always equal to two right angles.



For, if $\angle ACD = \angle BCD$, then each of them is a *right angle* (10), and $\therefore$ the two together make two right angles.

Or, if the angles ACD, BCD are unequal, from the point C in AB draw CE at right angles to AB (28); then, supposing ACD to be the greater of the two angles ACD, BCD, it is evident that $\angle ACD$ is as much greater than a right angle as BCD is less, and $\therefore$ that the two together are equal to *two* right angles.

COR. 1. If the two angles ACD, BCD, with the same *vertex* C, are together equal to two right angles, AC, and CB, are in one and the same straight line.

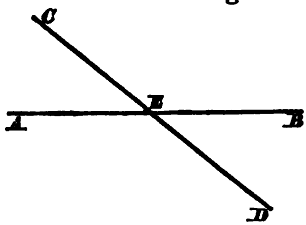
COR. 2. Hence, also, whatever be the number of angles in one plane, separate and distinct, on one side of AB, with a common vertex C, the *sum* of all these angles is equal to two right angles; and similarly on the other side of AB. So that all the angles in one plane exactly occupying the whole space round any given point are together equal to *four* right angles.

31. PROP. IX. *If one straight line intersects* another straight line, the vertical, or opposite, angles shall be equal to one another.*

Let the straight line AB intersect the straight line CD in the point E; then

$$\angle AEC = \angle BED,$$

and $\angle AED = \angle BEC$.



For, since CE meets AB, and makes with it the angles AEC, BEC, these are together equal to two right angles (30); and since BE meets CD, and makes with it the angles BEC, BED, these also are equal to two right angles; $\therefore$ the angles AEC, BEC together are equal to the angles BEC, and BED together; and, taking the same angle BEC from these equals, the remainders must be equal, that is,

$$\angle AEC = \angle BED.$$

Similarly, $\angle^s$ AEC, AED are equal to two right angles; and so also are $\angle^s$ AEC, BEC;

$$\therefore \angle AED = \angle BEC.$$

COR. Hence, if the lines forming any angle be 'produced', or extended, through the vertex in the opposite direction, the new angle thus formed will be equal to the other.

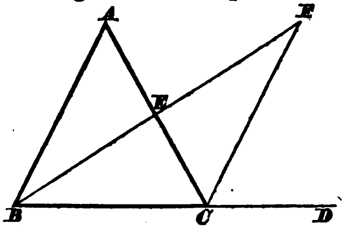
32. PROP. X. *If a side of a triangle be 'produced'†,*

* One line *intersects* another when it not only *meets* that other, but crosses it and is continued on the other side.

† That is, be *extended*, or continued indefinitely *in the same straight line.*

the exterior angle, thus formed by the adjacent side and the side produced, is greater than either of the interior opposite angles.

Let the side BC of the triangle ABC be 'produced' to any point D; the exterior angle ACD shall be *greater* than either of the interior opposite angles ABC, BAC.



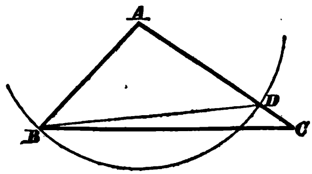
Bisect the side AC in the point E (27), join BE, produce it to F, making EF equal to EB. (by describing a circle with centre E and radius EB to cut BE produced in F), and join CF.

Then in the triangles AEB, CEF, AE, EB are equal to CE, EF, each to each, by construction, and $\angle AEB = \angle CEF$ (31), because they are opposite vertical angles, ∴ the triangles AEB, CEF are equal in all respects (24), and the angles are equal to which the equal sides are opposite, so that $\angle BAE = \angle ECF$; but $\angle ACD$ is greater than $\angle ECF$ (8), ∴ $\angle ACD$ is greater than $\angle BAE$, or $\angle BAC$, which is the same thing.

Similarly it may be shewn, by bisecting BC in G, joining AG, and proceeding as before, that $\angle ACD$ is greater than $\angle ABC$.

33. Prop. XI. *The greater side of every triangle is opposite to the greater angle.*

Let ABC be a triangle, of which the side AC is greater than the side AB; $\angle ABC$ shall be greater than $\angle BCA$.



With centre A and radius AB describe an arc of a circle cutting AC in the point D; and join BD, which will necessarily fall within the triangle ABC.

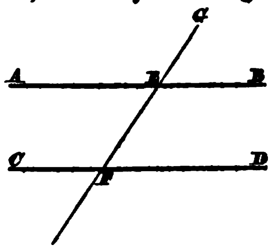
Then, since $AB = AD$, $\angle ABD = \angle ADB$ (26); but $\angle ADB$, the exterior angle of the triangle DBC, is greater than the interior opposite $\angle BCD$ (32); ∴ $\angle ABD$ is greater than $\angle BCD$, or $\angle BCA$, and ∴ *à fortiori* $\angle ABC$ is greater than $\angle BCA$.

COR. Conversely, the greater angle of every triangle is subtended by the greater side.

34. PROP. XII. *If a straight line, meeting two other straight lines in the same plane, form angles with each, on contrary sides of itself, equal to one another, these two straight lines shall be parallel.*

[The angles here described are, for shortness, called *alternate* angles.]

Let the straight lines AB, CD, be met by the straight line EF, so that $\angle AEF = \angle EFD$; or $\angle BEF = \angle CFE$; then AB is parallel to CD.



For, if AB, CD are *not* parallel, they will meet, upon being produced either towards B, D, or towards A, C. Suppose them to meet towards B, D, then a *triangle* will be formed of which EF is one *side*, and $\angle AEF$ will be the '*exterior*' angle of that triangle, mentioned in a former proposition (32), $\therefore \angle AEF$ is *greater than* the interior opposite angle EFD. But, by the supposition, $\angle AEF = \angle EFD$; and $\therefore$ if AB, CD meet *anywhere* towards B, D, $\angle AEF$ is both *greater than*, and *equal to*, $\angle EFD$; which is manifestly impossible. Hence AB, CD do *not* meet towards, B, D; and in the same manner it may be shewn that they do not meet towards A, C; $\therefore$ AB, CD are *parallel*, according to the Definition of parallel lines (12).

COR. 1. Produce FE to any point G; if the *exterior* angle, as $\angle BEG$, be equal to the interior and opposite angle on the same side of the intersecting line, as $\angle EFD$; then also AB shall be parallel to CD.

For $\angle BEG = \angle AEF$ (31), $\therefore$ $\angle AEF = \angle EFD$, and $\therefore$ AB, CD are parallel, as already proved.

COR. 2. If the two *interior* angles on the same side of the intersecting line are together equal to two right angles, the two straight lines shall be parallel.

For, since $\angle BEF + \angle BEG =$ two right angles (30),

$$\therefore \angle BEF + \angle BEG = \angle BEF + \angle EFD,$$

and $\therefore \angle BEG = \angle EFD$, $\therefore$ AB is parallel to CD, by COR. 1.

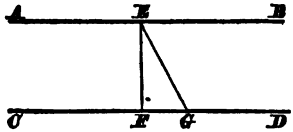
COR. 3. If two straight lines be parallel, and from

any point in one of them a straight line be drawn at right angles to that one, and produced to meet the other, it will also be at right angles to the other. This is obvious from Cor. 2.

Cor. 4. The converse of both the original proposition and each of the three preceding Corollaries also holds true, viz. that if two *parallel* straight lines be intersected by a third, the alternate angles are equal to one another; and the exterior angle is equal to the interior and opposite upon the same side; and also the two interior angles upon the same side are equal to two right angles.

35. Prop. XIII. *The straight line which joins two parallel straight lines, and is at right angles to each of them, is the shortest line which can be drawn from one to the other.*

Let AB, CD be two *parallel* straight lines, take any point E in AB, and draw EF at right angles to AB, meeting CD in F; EF shall be the *shortest* line which can be drawn from E to meet CD.

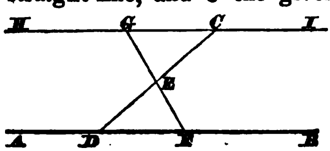


From E draw any other line EG to meet CD in some point G between F and D. Then $\angle BEG = \angle EGF$ (34), since they are '*alternate angles*'; and $\angle BEF$ is greater than $\angle BEG$, $\therefore \angle EFG$ is greater than $\angle EGF$. But the greater side is opposite to the greater angle, (33. Cor.) $\therefore$ EG is greater than EF, and EG is *any* other line than EF joining the parallels, $\therefore$ EF is the shortest of all such lines.

Cor. All lines joining *at right angles* the same two parallels are equal to each other. The most common and popular notion of *parallel* lines is based upon this property.

36. Prop. XIV. *To draw a straight line parallel to a given straight line through any proposed point without it.*

Let AB be the given straight line, and C the given point without it. It is required to draw, through the point C, a straight line parallel to AB.



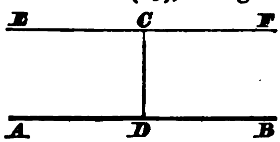
Take any point D in AB, and join CD; bi-

sect CD in the point E (27); from E draw EF to meet AB in any other point F; produce FE to G, making $EG = EF$ (by drawing a circle with centre E and radius EF); join the points G and C by the straight line GC, and produce GC both ways indefinitely to H and I; then HCI is a straight line through the point C parallel to AB.

For $ED = EC$, and $EF = EG$, $\therefore$ in the triangles EDF, ECG, the two sides ED, EF are equal to the two sides EC, EG; also $\angle DEF = \angle CEG$ (31), $\therefore$ the triangles are equal in all respects, and $\angle EDF = \angle ECG$, or, which is the same thing, $\angle CDB = \angle HCD$, and they are '*alternate angles*', $\therefore$ HCI is parallel to AB (34).

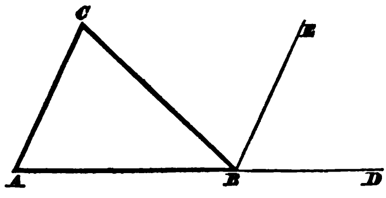
Another Method. The same thing may be done as follows:

From C draw CD *perpendicular* to AB (29), and again from C draw CE at right angles to CD (28); produce EC to any point F; then EF is parallel to AB and is drawn through C (34, Cor. 3).



37. PROP. XV. *If a side of any triangle be produced, the exterior angle (formed by the adjacent side and the side produced) is equal to the two interior opposite angles of the triangle; and the three interior angles of every triangle are equal to two right angles.*

Let the side AB of the triangle ABC be 'produced' to D; $\angle CBD$ shall be equal to the angles BAC, ACB taken together; and the three angles ABC, ACB, BAC, shall together be equal to two right angles.

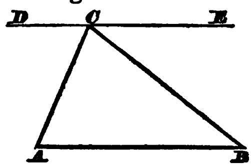


Through the point B draw BE parallel to AC (36); then because BC meets the two parallel straight lines AC, BE, $\angle EBC = \angle ACB$, and $\angle EBD = \angle BAC$, (34, Cor. 4); $\therefore$ $\angle CBD$, which is made up of $\angle EBC$, and $\angle EBD$, is equal to the two interior opposite angles BAC, ACB.

And since the angles *CBD*, *ABC* are equal to two right angles (30), and $\angle CBD$ is equal to the two angles *BAC*, *ACB*, $\therefore$ the angles *BAC*, *ACB*, *ABC*, are equal to two right angles.

Another Method. That the three angles of a triangle are equal to two right angles may be shewn in another very simple way thus:

Let *ABC* be the triangle, and through *C* draw *DCE* parallel to *AB* (36), then the angles *ACB*, *ACD*, *BCE* are together equal to two right angles (30. Cor. 2). But since *DE* is parallel to *AB*, $\angle ACD = \angle BAC$, and $\angle BCE = \angle ABC$ (34, Cor. 4), $\therefore$ the angles *BAC*, *ACB*, *ABC*, are equal to two right angles.

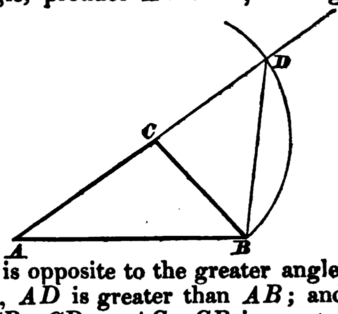


Cor. Hence no triangle can have more than *one* right angle, or *one* obtuse angle.

38. Prop. XVI. *Any two sides of a triangle are together greater than the third side.*

[If this be not evident from the fact that the straight line joining any two points is less than any crooked line joining the same points, the following proof may be given:]

Let *ABC* be a triangle, produce *AC* to *D*, making *CD* equal to *CB*, (by drawing a circle with centre *C* and radius *CB*); and join *BD*.



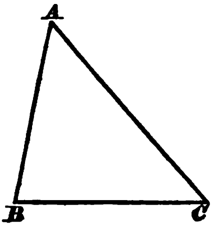
Then, since *CB*=*CD*, $\angle CBD = \angle CDB$ (26); but $\angle ABD$ is greater than $\angle CBD$, $\therefore$ $\angle ABD$ is greater than $\angle CDB$, or $\angle ADB$; and in every triangle the greater side is opposite to the greater angle, $\therefore$ in the triangle *ABD*, *AD* is greater than *AB*; and $AD = AC + CB$, since $CB = CD$, $\therefore$ $AC + CB$ is greater than *AB*.

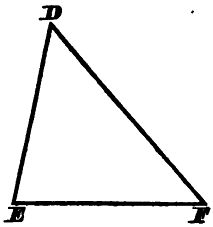
Similarly it may be shewn that $AB + BC$ is greater than *AC*, and $AC + AB$ greater than *BC*.

Cor. Hence, also, the *difference* between any two sides is *less than* the third side.

39. Prop. XVII. *If two triangles have two angles of the one equal to two angles of the other, each to each, and likewise the side which is common to those angles in the one equal to the side which is common to the two angles equal to them in the other, the triangles shall be equal in all respects.*

Let ABC, DEF, be two triangles, in which $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$, and side $BC =$ side EF; then the triangle ABC shall be equal to the triangle DEF in all respects.

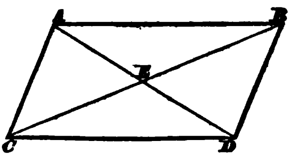




For, if the triangle ABC be 'applied to,' or laid upon, the triangle DEF, so that the point B shall be upon E, and the line BC upon EF, the point C will fall upon F, because $BC = EF$. Also the side BA will fall upon ED, because $\angle ABC = \angle DEF$; and CA will fall upon FD, because $\angle ACB = \angle DFE$ (8). Hence BA coinciding with ED, and CA with FD, the point A cannot but coincide with the point D; and $\therefore$ the triangles coincide, or are equal in all respects.

40. Prop. XVIII. *In every parallelogram the opposite sides are equal to one another; and so are the opposite angles. Likewise the diameter*, or diagonal, divides the parallelogram into two equal parts.*

Let $ACDB$ be a *parallelogram*, of which BC is a diameter, or diagonal; then $AB = CD$, $AC = BD$, $\angle ABD = \angle ACD$, $\angle BAC = \angle BDC$, and the triangle $ABC =$ the triangle BCD.



All that is here known and given is, that AB is *parallel* to CD, and AC *parallel* to BD (13).

Now since AB is parallel to CD, and BC meets them, $\angle ABC = \angle BCD$ (34, Cor. 4); and since AC is parallel

* The word '*diameter*' is seldom used in this meaning, being almost wholly restricted to the '*Circle*'.

to BD, and BC meets them, $\angle ACB = \angle DBC$; $\therefore$ in the two triangles ABC, BCD, two angles of the one are equal to two angles of the other, each to each, and one side BC, viz. the side common to those angles, the same in both, $\therefore$ the triangles are equal in all respects (39), and $\therefore$ $AB = CD$, $AC = BD$, and the triangle $ABC =$ the triangle BCD, that is, the diameter BC divides the parallelogram $ACDB$ into two equal parts.

Also, since $\angle ABC = \angle BCD$, and $\angle ACB = \angle DBC$, $\therefore$ $\angle ABC + \angle DBC$, or $\angle ABD = \angle BCD + \angle ACB$, or $\angle ACD$. And since the triangles ABC, BCD, are equal *in all respects*, $\therefore$ $\angle BAC = \angle BDC$.

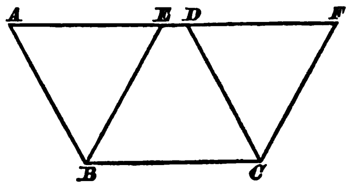
Cor. 1. Hence, *if any two adjacent sides of a parallelogram be equal to two adjacent sides of another parallelogram, each to each, and the angles contained by those sides are equal, the parallelograms will be equal in all respects.*

Cor. 2. *The two diagonals of every parallelogram bisect each other.*

Join AD, and let BC, AD, intersect in E; then, since $\angle ABE = \angle DCE$, and $\angle BAE = \angle CDE$, and side $AB =$ side CD, the triangles ABE, CDE, are equal *in all respects* (39); $\therefore$ side $AE =$ side DE. Similarly it may be shewn that $BE = CE$; therefore AD, BC bisect each other in the point E.

41. Prop. XIX. *All parallelograms, which have one side common and the sides opposite in one and the same straight line, are equal to one another*.*

Let $ABCD$, $EBCF$, be any two parallelograms, having the side BC common to both, and their opposite sides AD, EF in the same straight line $AEDF$; the parallelogram $ABCD$ shall be equal to the parallelogram $EBCF$.



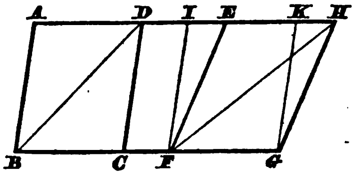
* Euclid's enunciation of this is '*Parallelograms upon the same base and between the same parallels are equal to one another*'. By '*base*' is meant the *side* on which the parallelogram may be supposed to *stand*; and '*between the same parallels*' means that the parallelograms are *bounded by* those parallels in *two* directions.

For $AD = BC = EF$ (40), $\therefore$ taking DE from each of these equals, $AE = DF$; also $EB = FC$ (40), and $\angle AEB = \angle CFD$ (34, Cor. 4), since BE, CF are parallel; $\therefore$ in the triangles AEB, DFC, the two sides AE, EB are equal to the two sides DF, FC, each to each, and $\angle AEB = \angle DFC$, $\therefore$ the triangles are equal in all respects (24), that is, the triangle AEB = the triangle DFC. Now, if these *equal* triangles be separately subtracted from the same area $ABCF$, the remainders must be equal, that is, the area $ABCD$ = the area $EBCF$.

[Observe, when D falls between A and E, DE must be *added*, instead of *subtracted*.]

Cor. 1. Hence, also, all parallelograms upon *equal* bases and 'between the same two parallels' are equal to one another.

For, if $ABCD$, $EFGH$, be two parallelograms upon *equal* bases BC, FG, and between the same parallels AH, BG, from the points F and G draw FI, GK, parallel to AB, or CD (36), meeting AH, or AH produced, in I and K. Then since $ABFI$ is a parallelogram, $FI = AB$ (40). And $FG = BC$; and $\angle GFI = \angle ABC$; $\therefore$ the parallelogram $FGKI$ = the parallelogram $ABCD$, (40. Cor. 1). But the parallelogram $EFGH$ = the parallelogram $FGKI$, as has been proved, $\therefore$ the parallelogram $EFGH$ = the parallelogram* $ABCD$.

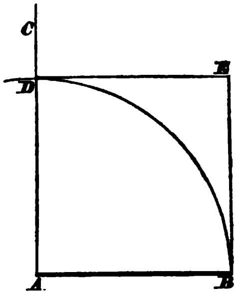


Cor. 2. Join BD, and FH, then BD is a diagonal of the parallelogram $ABCD$, and divides it into two equal parts (40); $\therefore$ the triangle BCD = half the parallelogram $ABCD$. Similarly the triangle FGH = half the parallelogram $EFGH$. And the halves of equal things must themselves be equal; $\therefore$ the triangle BCD = the triangle FGH; that is, *all triangles upon the same, or equal bases, and 'between the same parallels', are equal to one another.*

* '*Parallelogram*' being a long word, when it occurs frequently, it is allowable to abridge it thus, ▭^{m}. The word '*parallel*' also, in writing, is often abridged to ||. It is not advisable, however, *for the learner* to carry this sort of symbolism very far.

42. Prop. XX. *To describe or construct a square upon a given straight line, that is, so as to have the given straight line for one of its sides.*

Let AB be the given straight line. From the point A draw the indefinite straight line AC at right angles to AB (28); with centre A and radius AB describe an arc of a circle cutting AC in D; through D draw DE parallel to AB, and through B draw BE parallel to AD: then $ABED$ shall be a square.



For $AB = AD$; and $ABED$ is a *parallelogram*, which has its opposite sides equal, and also its opposite angles (40); $\therefore$ $AB = DE = AD = BE$, that is, all the sides of $ABCD$ are equal. Also, since $\angle BAD$ is a right angle, and AD is parallel to BE, $\therefore$ $\angle ABE$ is a right angle. But the opposite angles are equal, $\therefore$ *all* the angles of $ABED$ are right angles.

Hence, by the definition, $ABED$ is a square, and it is described upon the line AB.

Cor. 1. If the *side* of one square be equal to the side of another square, the squares are *equal* in all respects.

Cor. 2. If a parallelogram have *one* of its angles a right angle, it has *four* right angles.

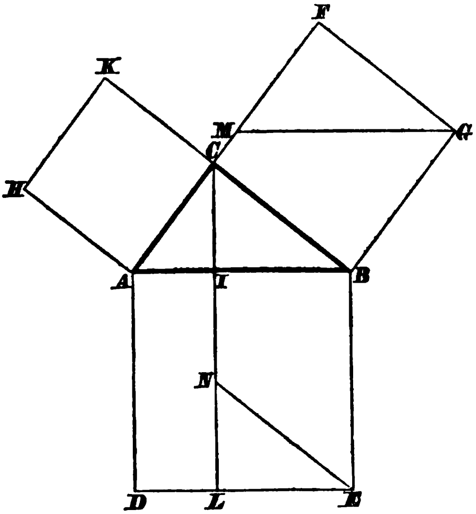
Cor. 3. If two squares be equal, their *sides* are equal.

43. Prop. XXI. *In the case of any right-angled triangle the square described upon the side opposite to the right angle* is equal to the two squares together described upon the two other sides which form the right angle.*

Let ABC be a right-angled triangle, in which $\angle ACB$ is the right angle. Upon the side AB, opposite to the right angle describe the square $ADEB$ (42); upon BC the square $BCFG$; and upon AC the square $AHKC$. Then the square $ADEB$ shall be equal to the two squares $BCFG$, and $AHKC$ taken together (21).

* The side of the triangle which is opposite to the *right angle* is sometimes called the '*hypothenuse*', from a Greek word signifying to *subtend* because it *subtends* the right angle.

From the point C draw the straight line CIL parallel to AD, meeting AB in I, and DE in L. Through G draw GM parallel to AB, meeting AC or CF in M; and through E draw EN parallel to BC, meeting CL in N.



Then, since $\angle ACB$ = a right angle $= \angle BCF$, ACF is a straight line (30. Cor. 1.), and it is parallel to BG, because $BCFG$ is a parallelogram. Also $BAMG$ is a parallelogram; therefore, since $BCFG$, $BAMG$, are parallelograms upon the same base BG, and between the same parallels AF, BG, they are equal to each other, that is, the square described on BC = the parallelogram $BAMG$.

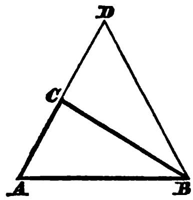
Again $AB = BE$, and $BG = BC$; also $\angle ABG = \angle ABC$ + a right angle $= \angle CBE$; therefore the two parallelograms $BAMG$, $BCNE$ have two adjacent sides of the one equal to two adjacent sides of the other, each to each, and likewise the angles between those sides equal; and $\therefore$ the parallelograms are equal, (40. Cor. 1.); that is, the parallelogram $BAMG$ = the parallelogram $BCNE$; $\therefore$ the square described on BC = the parallelogram $BCNE$ = the parallelogram $BILE$, which is on the same base BE, and between the same parallels BE, CL.

Similarly it may be shewn, by joining DN, and

drawing through H a line parallel to AB, that the square on AC = the parallelogram $AILD$; ∴ the square on BC + the square on AC = the parallelogram $BILE$ + the parallelogram $AILD$ = the square $ABED$ = square on AB.

N. B. The square described *upon* a line is generally called, for shortness, the square *of* that line. Thus the square described *upon* the line AB is called 'the square *of* AB.'

Cor. The converse of this proposition is also true, viz. that '*if the square described upon one of the sides of a triangle be equal to the sum of the squares described upon the other two sides of it, the angle between these two sides is a right angle.*'



Let ABC be a triangle, such that square of AC + square of BC = square of AB; from C draw CD at right angles to BC, making $CD = AC$; and join BD. Then, since $CD = AC$, the square of CD = square of AC, (42, Cor. 1) and square of CD + square of BC = square of AC + square of BC; but square of CD + square of BC = square of BD, because $\angle BCD$ is a *right* angle; and square of AC + square of BC = square of AB by the supposition; ∴ square of BD = square of AB, and ∴ $BD = AB$ (42, Cor. 3). Hence the two triangles ABC, BCD have all the sides of the one equal to the sides of the other, each to each, and ∴ the two triangles are equal, and their *angles* equal, each to each, to which the equal sides are opposite, ∴ $\angle ACB = \angle BCD$ = a right angle.

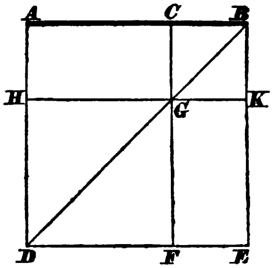
44. Prop. XXII. *If a straight line be divided into any two parts, the square of the whole line is equal to the sum of the squares of the two parts together with twice the rectangle* contained by the parts.*

[Def. A *rectangle* is said to be *contained* by any two of its adjacent *sides*†.]

* A *rectangle* has been already defined (13) as a *plane surface* in the form of a *parallelogram* with all its angles right angles. Care must be taken not to confound it with '*right angle*'.

† Since the opposite *sides* of every *parallelogram* and therefore of a *rectangle*, are equal to one another, and likewise the opposite *angles*,

Let AB be a straight line divided into any two parts in C; upon AB describe the square $ADEB$ (42); join BD; through the point C draw CGF parallel to AD or BE, and meeting BD in G; and through G draw HGK parallel to AB.



Then, since $BCGK$ is a parallelogram by construction, its opposite sides are equal to each other, and likewise its opposite angles (40), that is, $BC = KG$, $BK = CG$, $\angle CBK = \angle CGK$, and $\angle BCG = \angle BKG$. But since $BE = ED$, $\therefore \angle EBD = \angle EDB$ (26); and since KG is parallel to ED, $\angle KGB = \angle EDB$ (34), $\therefore \angle KBG$, which is the same as $\angle EBD$, $= \angle KGB$, and $\therefore BK = KG$; but $BC = KG$, and $BK = CG$, $\therefore BCGK$ is equilateral.

Again, since BC is parallel to KG, $\angle CBK + \angle BKG$ = *two* right angles (34); but $\angle CBK$ is a right angle, $\therefore$ $\angle BKG$ is also a right angle; and the opposite angles are equal, $\therefore BCGK$ has all its angles right angles. And it has been proved to have all its sides equal. It is therefore a *square;* and it is the square of BC.

Similarly it may be shewn, that $HGFD$ is a *square;* and it is the square of HG, or AC, since $ACGH$ is a parallelogram of which the side $AC = HG$.

Also, since $\angle BCG$ is a right angle, $\therefore \angle ACG$ is a right angle, and $\therefore$ the parallelogram $ACGH$ is a *rectangle,* and it is '*contained by*'* AC, CG, or by AC, CB, since $CB = CG$. And, similarly, $EFGK$ is a rectangle, contained by FG, GK, or by HG, GC, or by AC, CB. And these make up the whole area $ABED$, which is the square of AB; $\therefore$ the square of AB = the square of AC + the square BC + twice the rectangle AC, CB †.

(40), and since each of the angles of a *rectangle* is always a right angle, *two adjacent sides* alone will obviously serve to *fix* any rectangle; and hence it is, that the rectangle is said to be '*contained*' by those sides, because nothing more is needed to determine the rectangle.

* The expression '*contained by*' is mostly omitted; and the rectangle contained by any two lines, as AC, and CB, is simply called 'the rectangle AC, CB'.

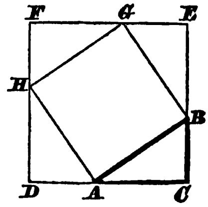
† By the help of this Proposition a very elegant proof of the important Theorem in (43) may be given as follows:—

45. Prop. XXIII. *If a straight line be divided into any two parts, the squares of the whole line and one of the parts are together equal to twice the rectangle contained by the whole and that part, together with the square of the other part.*

With the same construction as in (44), and in the same manner it may be shewn, that $BCGK$, and $HGFD$, are the *squares* of BC, and AC; and that $ACGH$ is a *rectangle*, and $= EFGK$, which is also a *rectangle*. Add to each of these equals the square $BCGK$, and then the rectangle $ABKH =$ the rectangle $EBCF$; $\therefore ABKH + EBCF =$ twice the rectangle $ABKH =$ twice the rectangle contained by AB, BC. Now add to these equals the square $HDFG$, which is the square of AC; then $ABKH + EBCF +$ square of $AC =$ twice the rectangle AB, $BC +$ square of AC. But the former of these equals make up the square $ABED +$ the square $BCGK$; $\therefore$ the squares of AB and BC are together equal to twice the rectangle AB, BC, together with the square of AC.

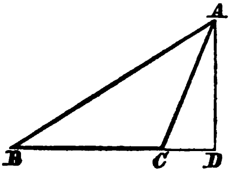
46. Prop. XXIV. *In any obtuse-angled triangle if a perpendicular be drawn from the vertex of either of the acute angles upon the opposite side produced, the square of the side subtending the obtuse angle is greater than the sum of the squares of the sides forming the obtuse angle by twice the rectangle contained by the side which is produced and the part produced,* viz. *the part intercepted between the perpendicular and the vertex of the obtuse angle.*

Let $\angle ACB$ be an *obtuse* angle of the triangle ABC; and from A draw AD perpendicular to BC produced;



Let ABC be a right-angled triangle; C the right angle; produce CA to D, making $AD = CB$. Upon CD describe the square $DCEF$. Take $EG = CB$, and $FH = CB$; and join BG, GH, HA. It may then easily be shewn, that $ABGH$ is a *square*, and that the four triangles ABC, BGE, GHF, HAD are equal to one another in all respects, so that the sum of them is equal to twice the rectangle AC, CB, or AC, AD, since $AD = CB$. Hence the square of $CD =$ the square of $AB +$ twice the rectangle AC, AD. But by (44) the square of $CD =$ the square of $AC +$ square of $AD +$ twice the rectangle AC, AD. Therefore the square of $AB =$ square of $AC +$ square of $AD =$ square of $AC +$ square of CB.

the square of AB shall be greater than the sum of the squares of AC, and BC, by twice the rectangle BC, CD.

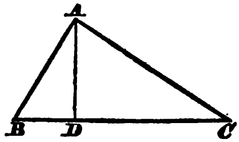


For the square of BD = the square of BC + the square of CD + twice the rectangle BC, CD (44). Add to these equals the square of AD; then the square of BD + square of AD = square of BC + square of CD + square of AD + twice the rectangle BC, CD. But (43) square BD + square of AD = square of AB; and square of CD + square of AD = square of AC; ∴ square of AB = square of BC + square of AC + twice the rectangle BC, CD; that is, square of AB is greater than the sum of the squares of AC, and BC, by twice the rectangle BC, CD.

47. Prop. XXV. *In every triangle the square of the side opposite to any acute angle is less than the sum of the squares of the sides forming that angle by twice the rectangle contained by either of these sides and that part of it which is intercepted between a perpendicular let fall upon it from the vertex of the opposite angle and the acute angle.*

Let ABC be an acute angle of the triangle ABC; from A draw AD perpendicular to BC meeting it in the point D. Then the square of AC shall be less than the squares of AB and BC by twice the rectangle BC, BD.

For by (45) the square of BC + the square of BD = twice the rectangle BC, BD + the square of CD. Add to each of these equals the square of AD; then the square of BC + the square of BD + the square of AD = twice the rectangle BC, BD + the square of CD + the square of AD. But, since $\angle ADB$ is a right angle, the squares of BD, and AD, are equal to the square of AB (43); also the squares of CD and AD are equal to the square of AC; ∴ the square of BC + the square of AB = twice the rectangle BC, BD + the square of AC; that is, the square of AC is less than the sum of the squares of AB, and BC by twice the rectangle BC, BD.



QUESTIONS AND EXERCISES IN THE PRECEDING PROPOSITIONS. A.

(1) In describing an equilateral triangle (23) upon a given straight line, how much is taken for granted? If a second triangle be drawn on the *opposite side* of the line by a similar construction, what figure will the two together make?

(2) Have we yet laid down any mode of *measuring* an angle? If not, how are we able to prove that one angle is *equal to*, *less than*, or *greater than*, another according to circumstances?

(3) If the *angles* of one triangle be equal to the angles of another, *each to each*, are the *triangles* necessarily equal? What is the force of the expression, '*each to each*'? Exhibit a case where the angles are respectively equal, but *not* '*each to each*'.

(4) What is meant in (25) by a '*given* angle'? Is it *necessary* that the triangle *DEF* should be *equilateral*? What other triangle would do as well? Does it matter on which side of *DE* the triangle is described?

(5) Define an '*isosceles*' triangle. Is an *equilateral* triangle isosceles? Can more than one isosceles triangle be constructed on the same base and on the same side of it? Can a *right-angled* triangle be also *isosceles*?

(6) What is the precise meaning of '*given* straight line' in (27), where it is required to *bisect* it? Is it the same as in (28) and (29)? If not, what is the difference?

(7) What is the meaning of the word '*base*' as applied to a triangle, and to a parallelogram? Is it restricted to one fixed side only?

(8) Shew that only *one* straight line can be drawn *perpendicular* to a given straight line from a given point without it.

(9) Shew that the *perpendicular* is the *shortest* of all lines from a given point to a given straight line. Of all such lines *which* measures the *distance* of the point from the given line?

(10) If in (30) straight lines be drawn bisecting

each of the angles ACD, BCD, shew that these straight lines are at right angles to one another.

(11) Shew that any point in the straight line bisecting an angle is equidistant from the two straight lines forming the angle.

(12) Shew that any side of a triangle is less than half the sum of all three sides of the same triangle.

(13) Shew that the straight line drawn from the middle point of the base of an isosceles triangle to the vertex of the opposite angle is at right angles to the base, and bisects the opposite angle.

(14) Shew that each angle of an equilateral triangle is *two-thirds* of a right angle. *Trisect* a right angle.

(15) Can a triangle have more than one of its angles a *right* angle, or an *obtuse* angle? If not, why not?

(16) Shew that the four angles of every quadrilateral figure are together equal to four right angles.

(17) If *one* of the angles of a *parallelogram* be a right angle, does this determine all the other angles?

(18) Shew that any two straight lines at right angles to the same straight line, and on the same side of it, are parallel.

(19) If two parallel straight lines be intersected by two other parallel straight lines, shew that the parts of the latter two intercepted *between* the former two are equal to each other.

(20) If two straight lines in the same plane be equal and parallel, shew that the straight lines joining their extremities towards the same parts are also equal and parallel.

(21) If two straight lines be drawn bisecting two angles of a triangle, shew that the point in which they intersect is equidistant from the three sides of the triangle.

(22) Is it correct to speak of drawing a line *from an angle?* The expression is found in Simson's Euclid; what does it mean? See definition of *angle*.

(23) Simson also, after defining '*vertex*' *of an angle*, on the first occasion of using the term (Prop. VII) speaks

of the 'vertex' *of a triangle.* What is the difference betwixt the two?

(24) Shew that the straight line drawn from the *vertex* of the right angle in a right-angled triangle to the middle point of the hypothenuse is equal to half the hypothenuse.

(25) Explain what is meant by *the square of a line.* Is the square of the line AB the same as the square of the line BA?

(26) Take the particular case of a *right-angled* triangle which is *isosceles,* and shew how the squares described on the two sides can be made to cover the square on the hypothenuse.

(27) Is the square of AB *double* of the square of *half* AB? If not, what then?

(28) Make a square which shall be *double* of the square of a given line.

(29) Is the *rectangle* contained by AB, BC, the same as that contained by CB, BA? or that contained by BC, AB?

(30) Make a rectangle which shall be *double* of a given rectangle AB, BC.

(31) Is it certain *à priori* that either a *square,* or an *equilateral* triangle, according to *Definition,* is *possible?* Explain fully.

(32) Make a *right-angled* triangle which shall be *double* of a given *right-angled* triangle.

(33) Can a triangle be equal to a rectangle? If so, draw a rectangle equal to a given triangle.

(34) Each of the sides of a rectangle is double of the corresponding side of another rectangle; how many times does the larger rectangle contain the other?

(35) If a side of an equilateral triangle be double of the side of another equilateral triangle, what proportion will the two triangles bear to each other?

(36) Shew that the *diagonals* of a square bisect each other at right angles.

(37) Shew that in every parallelogram the squares of the diagonals are together equal to the sum of the squares of all the sides.

THE CIRCLE AND STRAIGHT LINES CONNECTED WITH IT.

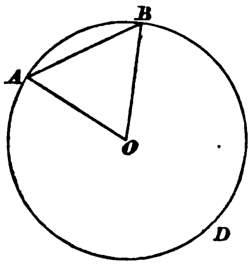
48. Definitions. An Arc of a circle is a *portion* of the *circumference* of the circle.

A Chord is the straight line which joins the two extremities of an *arc*.

A *segment* of a circle is a portion of a circle bounded by an *arc* and its *chord*.

A *sector* of a circle is a portion of a circle bounded by an *arc* and two *radii* drawn to the two extremities of the arc.

Thus, in the annexed fig. the *curved* line from A to B is an *arc*, the straight line AB is a *chord*, the area enclosed between the *arc* and the *chord* AB is a *segment*, and the area enclosed between the *arc* and the two radii OA, OB is a *sector*, of the circle ABD whose centre is O.



Hence a *diameter* is a particular *chord*; a *semi-circle* is a particular *segment*; and a *quadrant* is a particular *sector*.

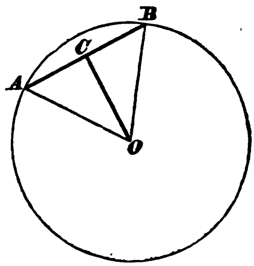
The learner must keep in mind the difference between *arc*, *segment*, and *sector*. Observe, that an *arc* is a *line*; but a *segment*, and a *sector*, are both *areas*.

Observe also that, according to the Definition, the straight line AB is the *chord* of the *arc* ADB as well as of the arc AB; but the *smaller* arc of the two is always meant except when it is otherwise expressed.

49. Prop. I. *A straight line drawn from the centre of a circle to the middle point of a chord is perpendicular to that chord.*

Let AB be the *chord* of any *arc* AB of a circle whose centre is O*; and C the middle point in the chord. Join OC; then OC shall be perpendicular to the chord AB.

* It is not necessary here to *determine* the precise position of the centre, but merely to *assume*, according to the definition, that there is such a point somewhere within the circle, and to *call* it the point O.



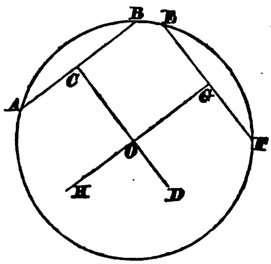
For, joining OA, OB, in the two triangles OAC, OBC, the two sides OA, AC, are equal to the two sides OB, BC, each to each; and $\angle OAC = \angle OBC$, since $OA = OB$ (26), $\therefore$ the triangles are equal in all respects (24), and $\therefore \angle OCA = \angle OCB$, and $\therefore$ each of them is a right angle; that is, OC is perpendicular to AB.

Cor. Conversely, if a straight line be drawn from the middle point of a chord at right angles to the chord, that straight line shall pass through the centre of the circle.

Also, a perpendicular drawn from the centre of a circle to a chord will bisect the chord.

50. Prop. II. *To find the centre of a given* circle.*

Let the annexed fig. be a given circle; and let it be required to find its centre.



Take any two points A, B, in the circumference, and join AB; bisect AB in C; and from C draw CD at right angles to AB. Then the centre of the circle is somewhere in the line CD (49). Again take two other points E, F in the circumference; join EF; bisect EF in G; and draw GH at right angles to EF, intersecting CD in the point O. Then the centre of the circle is in GH; and it is also in CD; but CD and GH have only one point in common, viz. the point O; $\therefore O$ is the centre of the circle.

Cor. The same method evidently applies to the case of a given *segment*, or *arc*, when the centre of the circle to which it belongs is required, or when it is required to *complete* the circle.

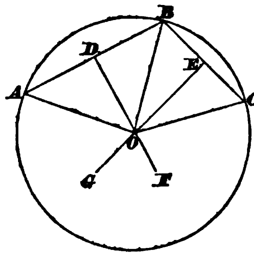
Another Method. Draw the chord AB; bisect it in

* By a *given* circle is here meant a plane surface presented to us in the form of a circle, as a *crown-piece*, or the end of a round ruler. Or it is a circle whose circumference is *traced out upon a plane surface.*

C; draw CD at right angles to AB, and meeting the circumference in D; produce DC to meet the circumference again in E. Then bisect DE in O; and since DE is a *diameter* (49 Cor.), $\therefore$ O is the centre of the circle.

51. Prop. III. *To describe a circle whose circumference shall pass through three given points.*

Let A, B, C, be the three given points; join AB, and BC. Bisect AB in D, and BC in E; from D draw DF at right angles to AB; and from E draw EG at right angles to BC, intersecting DF in O; with centre O and radius OA describe a circle, and its circumference shall pass through A, B, and C.



Join AO, BO, CO; then in the triangles ADO, BDO, $AD = BD$, $\therefore$ AD, DO are equal to BD, DO, each to each, and $\angle ADO = \angle BDO$, $\therefore$ $AO = BO$ (24). In the same way it may be shewn that $BO = CO$; $\therefore$ $AO = BO = CO$; that is, a circle described with centre O and radius AO, will pass through the points A, B, C.

N. B. If the given points A, B, C be in one and the same straight line, this construction will fail, because then DF and EG being at right angles to the same straight line will be *parallel* to each other, and never meet at all in O. In this particular case there is no circle whose circumference can be made to pass through the three given points*.

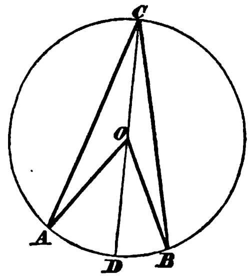
Also, if it be required to describe a circle whose circumference shall pass through *two* given points, A, B, it is plain, that there will be an *infinite number* of such circles, having their centres in the indefinite straight line DF.

52. Prop. IV. *The angle which any arc of a circle subtends at the centre of the circle is double of the angle which it subtends at the circumference†.*

* From this it follows that no straight line can meet the circumference of a circle in more than *two* points.

† The angle which an arc, or other magnitude, *subtends at a given*

Let AB be any arc of a circle whose centre is O; C *any* point in the other portion of the circumference. Join AO, BO, AC, BC; then $\angle AOB$, which the arc AB subtends at O, shall be double of $\angle ACB$ which it subtends at C.



Join CO, and produce it to meet the circumference in D; then since $OA = OC$, $\angle OAC = \angle OCA$ (26); and $\angle AOD = \angle OAC + \angle OCA$ (37) = twice $\angle OCA$.

Similarly $\angle BOD =$ twice $\angle OCB$;

$$\therefore \angle AOB = \angle AOD + \angle BOD \text{ (22)},$$
$$= \text{twice } \angle OCA + \text{twice } \angle OCB,$$
$$= \text{twice } \angle ACB.$$

If the point C be taken so that the centre O falls *without* the $\angle ACB$, the construction is the same, and the proof also, except that angles are *subtracted* instead of *added* (22).

COR. Since $\angle AOB$ in the same circle is always the same for a given arc AB, it follows that $\angle ACB$, which is half of $\angle AOB$, is the same whatever point C in the circumference be taken; that is, if E be *any* other point in the circumference, and AE, BE be joined, $\angle AEB = \angle ACB$. These latter angles are, for shortness, called angles '*in a segment*'; and thus, with Euclid, we say, '*all angles in the same segment are equal to one another*'.

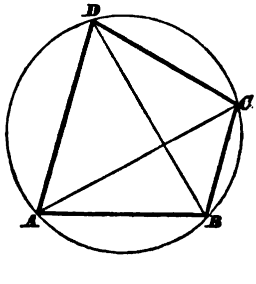
It is important for the student to make himself quite sure of the meaning of the phrase '*angle in a segment*'. In the 1st place, a *segment*, as already defined, is a portion of a circle bounded by an *arc* and the *chord* of the arc. Then an '*angle in the segment*' is the angle which that *chord subtends* at any point in the *arc*.

53. PROP. V. *In any four-sided rectilineal figure, which has all its angular points in the circumference of the same circle*, each pair of opposite angles is equal to two right angles.*

point, means the angle formed by two straight lines joining that point and the extreme points of the arc, or other magnitude.

* This is what is meant by the expression, sometimes used, '*a quadrilateral inscribed in a circle*'. To be *inscribed* it is necessary, that *all* the *angular points* fall upon the *circumference* of the circle.

Let $ABCD$ be a four-sided plane figure, having its angular points A, B, C, D, in the circumference of a circle; then $\angle ABC + \angle ADC =$ two right angles; and likewise $\angle BAD + \angle BCD =$ two right angles.



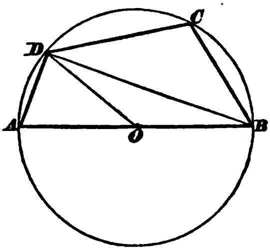
Join AC, BD; then in the triangle ABC, $\angle ABC + \angle BAC + \angle ACB =$ two right angles (37); but by (52) $\angle BAC = \angle BDC$, being angles '*in the same segment*' $BADC$. Also $\angle ACB = \angle ADB$, being angles '*in the same segment*' $ADCB$; $\therefore \angle ABC + \angle BDC + \angle ADB =$ two right angles; and $\angle BDC + \angle ADB = \angle ADC$ (22), $\therefore \angle ABC + \angle ADC =$ two right angles.

In the same manner it may be shewn that $\angle BAD + \angle BCD =$ two right angles.

COR. Hence no *parallelogram* except a *rectangle* can be '*inscribed*' in a circle; because the opposite angles in every parallelogram are equal to one another; and therefore in this case each of them must be a *right angle*.

54. PROP. VI. *The 'angle in a segment' equal to a semi-circle is a right angle; in a segment greater than a semi-circle is less than a right angle; and in a segment less than a semi-circle is greater than a right angle.*

Let $ABCD$ be a circle of which AB is a diameter, and O the centre; draw the chord BD dividing the circle into two segments, viz. BAD greater than, and BCD less than, a semi-circle; join AD, DC, CB. Then $\angle ADB$ 'in a semi-circle' is a right angle; $\angle BAD$ 'in a segment' greater than a semi-circle is less than a right angle, and $\angle BCD$ 'in a segment' less than a semi-circle is greater than a right angle.



Join OD; then since $OA = OD$, $\angle OAD = \angle ODA$ (26); and since $OB = OD$, $\angle OBD = \angle ODB$, $\therefore \angle ADB = \angle OAD + \angle OBD$; add to these equals $\angle ADB$, then twice $\angle ADB$

$= \angle OAD + \angle OBD + \angle ADB$; $\therefore$ twice $\angle ADB =$ two right angles, (since $\angle OAD$, $\angle OBD$, and $\angle ADB$ are the three angles of the triangle ABD), and the halves of equal things must be equal, $\therefore$ $\angle ADB =$ one right angle.

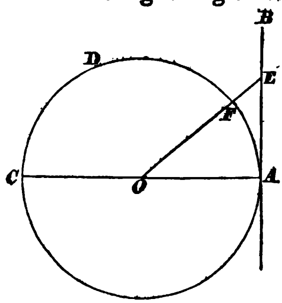
Also, since $\angle OAD + \angle OBD =$ a right angle, $\therefore \angle OAD$, or $\angle BAD$, which is the same thing, is *less than* a right angle.

Again, by (53), $\angle BCD + \angle BAD =$ two right angles, and since $\angle BAD$ is *less than* a right angle, $\therefore$ $\angle BCD$ must be *greater than* a right angle.

55. Prop. VII. *A straight line, drawn at right angles to a diameter of a circle from either of its extremities, lies wholly without the circumference of the circle except at that point only.*

[Def. A straight line, which lies wholly without the circumference of a circle except at one point only, is said to *touch* the circle, or to be a *tangent* to the circle, at that point.]

Let a straight line AB be drawn at right angles to AC, a diameter of the circle ACD, from the point A one of its extremities; and let O be the centre of the circle. Take E *any* other point in AB, distinct from A, and join OE; and let OE, or OE produced meet the circumference in F. Then AOE is a triangle; and since the three angles of every triangle are equal to two right angles,



and one of them in this case, viz. $\angle OAE$ is a right angle, $\therefore$ each of the other two angles, $\angle OEA$, $\angle AOE$, is *less than* a right angle; that is, $\angle OAE$ is greater than $\angle OEA$. But the greater side is opposite to the greater angle (33); $\therefore$ OE is greater than OA, the radius of the circle, that is, OE is greater than OF, or E is *without* the circumference. And E is *any* point whatever in AB except A; $\therefore$ AB lies wholly without the circumference except at the point A.

Cor. 1. The *converse* of this is also true, viz. that, if a straight line '*touch*' the circle at any point, it will be at right angles to the diameter or radius through that

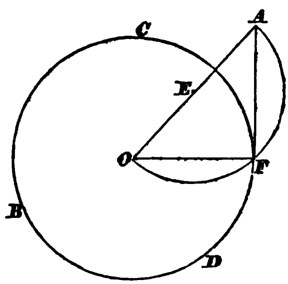
point. For, if possible, AB being a *tangent* at A, that is, every point in it except A being without the circle, suppose $\angle OAB$ *not* a right angle. From O draw OE at right angles to AB, meeting the circumference in F; then OAE is a *triangle*, of which $\angle OEA$ is a right angle, and $\therefore \angle OAE$ less than a right angle; $\therefore$ the side OA is greater than the side OE (33); but $OA = OF$, $\therefore OF$ is greater than OE, a part greater than the whole, which is impossible. Hence the supposition that $\angle OAB$ is *not* a right angle cannot hold; $\therefore AB$ must be at right angles to AO.

COR. 2. Hence, to draw a *tangent* to a given circle through a given point A in its circumference, find O the centre of the circle, join OA, and draw AB at right angles to OA from the point A; AB is the tangent required.

COR. 3. Hence, also, if a straight line *touches* a circle, and from the point of contact another straight line be drawn, at right angles to the former, through the circle, the centre of the circle will be in this latter line.

56. PROP. VIII. *To draw a tangent to a given circle from a given point without it.*

Let A be the given point from which it is required to draw a straight line *touching* the given circle BCD.



Find O the centre of the circle (50), and join OA; bisect OA in the point E; with centre E and radius EA describe the semi-circle AFO meeting the circle BCD in F; and join AF. AF is the tangent required.

For, joining OF, since AFO is a semi-circle, $\angle OFA$ is a right angle (54), $\therefore AF$ is at right angles to a radius or diameter of the circle from one of its extremities, $\therefore AF$ *touches* the circle at the point F (55).

A second *tangent* may also be drawn from A by

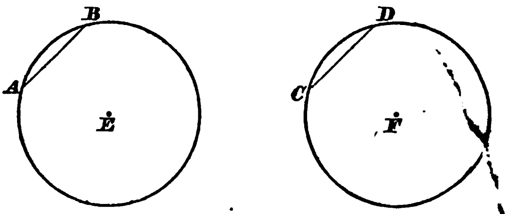
describing the semi-circle on the other side of OA meeting the given circle in C, and then joining AC.

57. Prop. IX. *If the radius of one circle be equal to the radius of another, the circles shall be equal in all respects.*

For, if one of the circles be 'applied to', or laid upon, the other so that their centres coincide, since the radii are equal, it is plain that the circumferences will coincide throughout; and, since the circumferences coincide in every part, it is evident that they enclose the same *area*, that is, the *circles* are equal to one another.

58. Prop. X. *In the same circle, or in equal circles, equal arcs have equal chords; and conversely, equal chords have equal arcs.*

Let AB, CD be equal *arcs* of two equal circles; draw the *chords* AB, CD; then chord AB = chord CD. For,



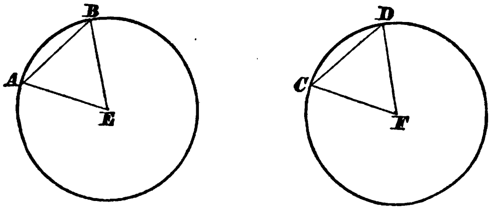
find the centres E, F of the circles, and suppose the one circle to be laid upon the other, so that the centre E shall be upon F; then since the radii are equal, the *whole* circumference of one will coincide with the *whole* circumference of the other; and, without altering this coincidence, if one of them be turned round the centre, in its own plane, until the point A coincides with the point C, the point B will coincide with D, because the *arc* AB = *arc* CD. So then, since the point A falls upon C, and B upon D, the *straight line* joining A and B must coincide with that joining C and D; that is, *chord* AB = *chord* CD.

And what is proved of *equal* circles will obviously hold true for equal arcs of the same circle.

And the converse also evidently follows, viz. that equal chords in the same circle, or in equal circles, subtend equal arcs.

59. Prop. XI. *In the same circle, or in equal circles, equal arcs subtend equal angles at the centre.*

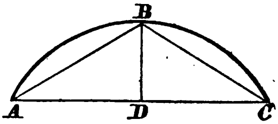
Let AB, CD be equal *arcs* of two equal circles, whose centres are E and F. Join AE, BE, CF, DF; then $\angle AEB = \angle CFD$.



For, joining the *chords* AB, CD, by (58) *chord* AB = *chord* CD; also $AE = CF$, and $BE = DF$; $\therefore$ the two triangles AEB, CFD, have all the sides of the one equal to all the sides of the other, each to each, and $\therefore$ the triangles are equal in all respects (23 Cor.). Consequently $\angle AEB = \angle CDF$, being the angles opposite to the equal sides AB, CD.

60. Prop. XII. *To bisect a given arc of a circle.*

Let ABC be the given arc. It is required to divide it into two parts in the point B, so that the *arc* AB = *arc* BC.



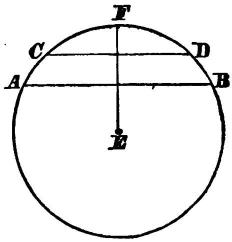
Join AC; bisect AC in D; from D draw DB at right angles to AC, intersecting the given arc in B. Then ABC is bisected in B.

For, drawing the *chords* AB, BC, the two sides AD, DB, in the triangle ADB, are equal to the two sides CD, DB, in the triangle CDB; also $\angle ADB = \angle CDB$, since each of them is a right angle, $\therefore$ the third side $AB = CB$. But equal chords subtend equal arcs, (58), $\therefore$ *arc* AB = *arc* BC.

Cor. If BD be produced, it will pass through the *centre* of the circle. And, conversely, the line drawn from the centre, bisecting the *chord*, will also bisect the *arc*.

61. Prop. XIII. *Two parallel chords in any circle will intercept equal arcs.*

Let AB, CD be any two *parallel* chords in the same circle; the arc AC shall be equal to the arc BD.



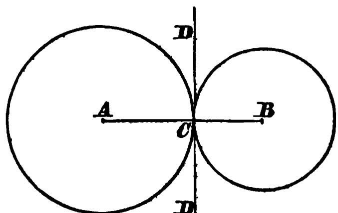
Find E the centre of the circle; and draw EF perpendicular to AB, meeting the circumference of the circle in F.

Then since CD is parallel to AB, EF is also perpendicular to CD (34 Cor. 3); ∴ both the chords AB, CD are bisected by the straight line EF (49 Cor.): and ∴ both the *arcs* AFB, CFD, are bisected in F (60); that is, arc AF = arc BF, and arc CF = arc DF; but if equals be taken from equals the remainders will be equal, ∴ arc AC = arc BD.

Conversely, if arc AC = arc BD, the chord AB is parallel to the chord CD.

62. Prop. XIV. *If the distance between the centres of two circles, which are in the same plane, be equal to the sum or difference of their radii, the circles will touch each other at one point only; and the point of contact will be in the straight line which joins the centres, or in that line produced.*

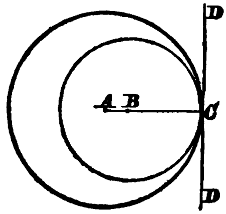
1. Let A, and B be the centres of two circles so situated, *in the same plane*, that AB, the straight line joining the centres is equal to the *sum* of their radii.



Let C be the point in which AB meets the circumference of the first circle, then AC = the radius of that circle; and since $AC + BC$ = the sum of the radii, BC must be the radius of the other circle; and ∴ C is a point in its circumference; that is, the two circumferences have the point C common to both. And they have no other point common: for, if CD be

drawn from C at right angles to AB, since CD is a tangent to both circles at the point C, every point in it, except the point C, is without both, that is, no point but C is common to the two, and ∴ they *touch* each other in that point.

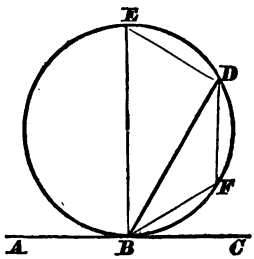
2. Let AB, the straight line joining the centres of the two circles, be equal to the *difference* of the radii. Produce AB to meet the circumference of the greater circle, whose centre is A, in C; then since AC is the radius of the greater circle, and AB is the difference of the two radii, BC = the radius of the smaller, and ∴ C is a point in the circumference of the latter; that is, C is a point common to both circumferences. That C is the only point common to the two circumferences is shewn precisely as in the former case; and ∴ the circles *touch* each other at that point.



In the former case the circles are said to touch each other *externally*, in the latter *internally*.

63. Prop. XV. *If a straight line touch a circle, and from the point of contact a chord be drawn dividing the circle into two segments, the angle between the tangent and this chord shall be equal to the angle 'in the alternate segment'* of the circle.*

Let the straight line ABC touch the circle BDE in the point B; and let BD be a *chord* dividing the circle into two segments. From B draw BE at right angles to AB, to meet the circumference again in E, which will ∴ be a *diameter* of the circle (55 Cor. 3). Join DE; take any point F in the arc BD, and join BF, DF. Then $\angle CBD = \angle BED$ 'in the alternate segment'; and $\angle ABD = \angle BFD$.



* By *alternate* segment is meant the segment *on the other side* of the chord.

For, since BE is a *diameter*, $\angle BDE$ is a *right angle* (54), being the angle 'in a semicircle'; $\therefore \angle BED + \angle EBD =$ a right angle (37) $= \angle CBD + \angle EBD$, $\therefore \angle CBD = \angle BED$.

Again, since $BFDE$ is a 'quadrilateral inscribed in a circle', $\angle BFD + \angle BED =$ two right angles (53) $= \angle ABD + \angle CBD$; and $\angle CBD$ has been shewn to be equal to $\angle BED$, $\therefore \angle ABD = \angle BFD$.

[It might appear, at first sight, that by drawing BE at *right angles* to AB, we have proved only a particular case of the proposition; but it is not so, because $\angle BED =$ *every other* angle '*in the same segment*' (52 Cor).]

EXERCISES B.

(1) Are all *diameters* of the same circle *equal* to one another? Shew that the *diameter* is greater than any other straight line drawn in the circle and terminated by the circumference.

(2) Does the *chord* of an arc *increase* as the arc increases? State the limitations.

(3) Can a circle be made up of *segments*? If so, of how many?

(4) Can a circle be made up of *sectors*? If so, of how many? In what case will a *sector* become a *segment*?

(5) Shew that the circumferences of circles which have the same centre cannot *cut* each other.

(6) If the circumference of a circle be divided into four equal arcs, shew that the *chords* of any two of them, which are adjacent, are at right angles to each other.

(7) If the circumference of a circle be divided into *six* equal parts, shew that the chord of each of them is equal to the radius.

(8) If the radius of a given circle be equal to a given straight line, find the *centre* of the circle.

(9) Make a circle of given radius, whose circumference shall pass through, 1st, *one* given point, 2ndly, *two* given points.

(10) Can more than *one* circle be drawn whose circumference shall pass through *three* given points?

(11) Shew that in *particular* cases a circle may be

drawn with its circumference passing through *four, or a greater number of,* given points. Exhibit such a case.

(12) In a given circle draw a chord which shall be both equal and parallel to a given chord in the same circle.

(13) If an arc or a segment of a circle be given, complete the circle.

(14) Through a *given point* within a given circle draw the *least* chord.

(15) Through a given point within a given circle draw a *chord* which shall be equal to a given line not greater than the diameter of the circle.

(16) If one circle intersect another, shew that the straight line joining the points of intersection is at right angles to the straight line joining their centres.

(17) Shew that the two tangents, which can be drawn to a circle from a point without it, are equal to one another.

(18) Shew that the straight line drawn through the middle point of an arc parallel to the *chord* of the arc is a *tangent* to the circle at that point.

(19) Can two distinct straight lines *touch* a circle at the same point?

(20) Divide a given arc into *four* equal parts.

(21) Have *equal* circles *equal* circumferences or perimeters? Is this the case with *equal* squares, triangles, and other rectilineal *equal* plane figures?

(22) If AB, CD be *any* two *chords* in a circle at right angles to each other, prove that the sum of the arcs AC, BD is equal to half the circumference.

(23) From two given points draw two straight lines which shall meet in a given straight line, and be at right angles to each other. Within what limits only is this possible?

(24) Apply Prop. VI. to draw a straight line at right angles to a given straight line from one extremity of it, when the given line cannot be produced'.

(25) Construct a square, when the diagonal only is given.

(26) If tangents be drawn to a circle from the extremities of a diameter, shew that any line intercepted between them, and touching the circle, subtends at the centre a right angle.

(27) Shew that about a given circle a certain number of equal circles can be drawn touching it and each other; and find the number.

(28) If two circles *touch* each other *internally*, and the radius of one of them be half that of the other, shew that every straight line, drawn from the point of contact to meet the outer circumference, is bisected by the inner one.

(29) A straight line touches a circle, and from the point of contact A any chord AB is drawn; BC is another chord parallel to the tangent, and BD a chord parallel to AC. Shew that the chords AB, AC, CD are equal to one another.

(30) If the circumferences of two circles intersect each other, and through one of the points of intersection the diameters be drawn, shew that the other extremities of those diameters and the other point of intersection will be in one and the same straight line.

(31) Three equal circles are given, in the same plane, of which no two intersect each other, find the point from which if tangents be drawn to each circle, those tangents shall be equal to one another.

(32) What is the angle which the arc of a quadrant subtends at any point in the remaining portion of the circumference? Is it the same for all circles?

(33) In any two circles which have the same centre *, if a *chord* be drawn to the outer one and intersecting the inner one, shew that the parts of this chord intercepted between the two *circumferences* are always equal.

(34) If two circles touch each other, either externally or internally, and through the point of contact two straight lines be drawn forming four chords, two in each circle, shew that the straight lines joining the extremities of these chords in each circle are *parallel* to one another.

* Such circles are sometimes called '*concentric*' circles.

PROPORTIONAL LINES AND AREAS.

64. Definition. *Ratio* is the relation which two or more things, or quantities of things, *of the same kind* bear *to each other in respect of magnitude.* And, for the purpose of this comparison, any two things are *of the same kind* only when the lesser of the two by multiplication can be made to exceed the other.

Thus a lineal foot can be multiplied (22) until it exceed a lineal mile; therefore these are things *of the same kind*, and bear a certain *ratio* to each other. So likewise an oz. and a lb. in weight have a certain *ratio;* a quart and a gallon have a certain *ratio;* and so on.

But an *oz.* and a *mile* are not things *of the same kind.* The one can never by multiplication be made to exceed the other; and consequently they bear no relation to each other *in respect of magnitude,* that is, they can have no *ratio* to each other.

Similarly, a *line* may have *ratio* to a *line,* and an *area* to an *area;* but a *line* can have no *ratio* to an *area,* because by the multiplication of either we can never arrive at, or exceed, the other.

65. Def. The *measure* of the *ratio* between any two magnitudes is, (not their *difference,* but) the number of *times* the one *contains,* or *is contained in,* the other.

Thus, if the line AB, upon being multiplied three times (22), becomes equal to the line CD, that is, if CD *contains* AB exactly three times, then the *measure* of the *ratio* of CD to AB is 3, that is, CD bears the same relation to AB *in magnitude* which 3 does to 1.

But in order that two magnitudes of the same kind may have a *ratio* to each other, it is not necessary that one should contain the other an *exact integral* number of times.

Thus, for example, let A be a magnitude which contains another magnitude taken as the *unit* of measurement, whatever that may be, 5 times; and let B be another magnitude, of the same kind, containing *the same unit* 3 times; then the *ratio* of A to B will be that of 5 to 3. In this case A may be said to *contain* B once and two-thirds of a time; and the *measure* of the *ratio* of A to B is $1\frac{2}{3}$, or the fraction $\frac{5}{3}$. Similarly in other cases.

In the case here supposed, a certain *multiple* of A is *equal* to another certain *multiple* of B, that is, *three* times

$A =$ *five* times B. Thus, if A be a line which contains a lineal foot 5 times, and B another line which contains it 3 times, then $A = 5$ feet, and $B = 3$ feet; the *ratio* of A to B is that of 5 feet to 3 feet, that is, 5 to 3; and 3 times $A = 15$ feet $= 5$ times B*.

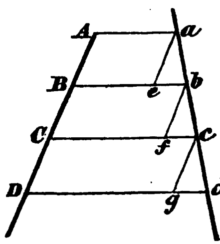
66. Def. Proportion is the *equality of ratios*. Thus, if the *ratio* of A to B be *equal* to the *ratio* of C to D, then A, B, C, D are said to be *proportionals*, or *in proportion*.

Observe, A, B, C, D, in order to be *proportionals*, need not be *all of the same kind*. It is only necessary that A and B be of the same kind, and likewise C and D of the same kind; but the one *pair* of magnitudes may be different from the other *pair*. Thus, one *line*, A, may have the same ratio to another *line*, B, that one *area*, C, has to another *area*, D, in which case A, B, C, D are *proportionals*.

The *ratio* of two magnitudes is often expressed by placing the symbol : between them; thus $A : B$ signifies the *ratio* of A to B. So then, if A, B, C, D are *proportionals*, $A : B = C : D$; but this is generally written thus, $A : B :: C : D$; and is read '*A is to B as C to D*', which means that A has the same *ratio* to B which C has to D.

67. Prop. I. *If two straight lines be intersected by any number of parallel lines, so that the parts of one of them intercepted between the parallels are equal to one another, the parts also, of the other line between the same parallels shall be equal to one another.*

Let $ABCD$ be any straight line, such that $AB = BC = CD$; or similarly, whatever the number of parts may be of which it is composed. Through the points A, B, C draw parallel lines Aa†, Bb, Cc, Dd, meeting another straight line in the points a, b, c, d; then also $ab = bc = cd$.



Through the point a draw ae parallel to AB, meeting Bb in e; and through b draw bf parallel

* In this section *single letters* will often be used to denote *lines* and *other magnitudes*, to avoid superfluous writing, where it may be done without risk of error.

† This is read '*A little* a,' '*B little* b', &c.

to BC meeting Cc in f. Then since $ABea$, and $BCfb$ are parallelograms, $ae = AB$, and $bf = BC$ (40); but $AB = BC$, $\therefore$ $ae = bf$.

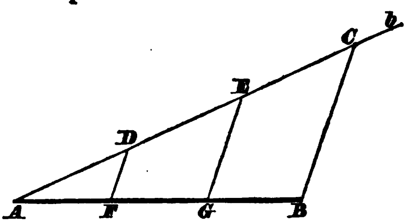
Again, because Bb is parallel to Cc, $\angle ABe = \angle BCf$; and because ae is parallel to AB, $\angle ABe = \angle aeb$; also because bf is parallel to BC, $\angle BCf = \angle bfc$, $\therefore$ $\angle aeb = \angle bfc$. And since ae, bf are parallel to the same line ABC, they are parallel to each other, and $\therefore$ $\angle eab = \angle fbc$. Hence in the two triangles aeb, bfc, there are two angles in the one equal to two angles in the other, each to each, and the side common to those angles in the one equal to the side which is common to the two angles, equal to them, in the other, $\therefore$ the triangles are equal in all respects, and the side ab = the side bc (39). Similarly it may be shewn, by drawing cg parallel to CD, that $bc = cd$; $\therefore$ $ab = bc = cd$. And the proof may be extended to *any number* of parts.

COR. 1. The proof here given is independent of the length of the line Aa, and will therefore hold when A and a coincide in the same point, that is, when the two given lines *meet* in A.

COR. 2. *Conversely*, if two straight lines be composed of the same number of equal parts, the straight lines Aa, Bb, Cc, &c., joining corresponding points in them, will be *parallel*.

68. PROP. II. *To divide a given straight line into any number of equal parts.*

Let AB be the given straight line, which is to be divided into any proposed number of equal parts, *three* suppose, as the process is the same whatever the number may be.

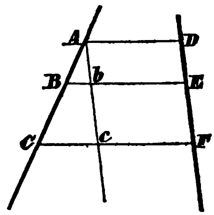


From A draw any other indefinite line Ab, forming an angle with AB; in Ab take any point D conveniently near to A, and with centre D and radius DA

describe a circle cutting *Ab* in *E*; then with centre *E* and the same radius as before describe a circle cutting *Ab* in *C*; so that $AD = DE = EC$. Join *CB*, and through *E* and *D* draw *EG* and *DF* parallel to *BC*, cutting *AB* in *G* and *F*. Then since $AD = DE = EC$, and *CB*, *EG*, *DF* are *parallels*, ∴ also $AF = FG = GB$, that is, *AB* is divided into three equal parts in the points *F* and *G*, (67 Cor. 1.)

69. Prop. III. *If any two straight lines be cut by three parallel straight lines, the parts intercepted between the parallels shall be 'proportionals'.*

Let *ABC*, *DEF* be any two straight lines, and *AD*, *BE*, *CF*, three parallel straight lines intersecting the former in *A*, *B*, *C*, and *D*, *E*, *F*. Then *AB*, *BC*, *DE*, *EF* shall be '*proportionals*', that is, *AB* shall have the same *ratio* to *BC* that *DE* has to *EF*.



Let *AB* contain the line which is the unit of measurement *three* times, and *BC* the same unit five times, (the *same* proof will hold for *any* numbers), and divide *AB* into three equal parts, and *BC* into five (68), and through the points of division draw lines parallel to *BE* intersecting *DE*, and *EF*: then *DE* will be divided by these *parallels* into the same number of *equal* parts as *AB*, and *EF* into the same number as *BC* (67), that is, *DE* will contain a certain line three times, and *EF the same line* five times; or the *ratio* of *DE* to *EF* is three to five, which is the same *ratio* as *AB* to *BC*; ∴ *AB*, *BC*, *DE*, *EF* are *proportionals* (66).

Observe, it may be that a *specified* unit of measurement will not *exactly* divide *AB*, and *BC*, in which case the unit must be *reduced*, until this can take place. For example, if there be not an exact number of *feet* in *AB*, and *BC*, there may be an exact number of *inches*; or, if not *inches*, there may be an exact number of *tenths* of an inch; and so on. And *whatever* be the reduced unit which will exactly divide *both AB and BC*, the proof above given then holds.

70. Prop. IV. *If two sides of a triangle be intersected by a straight line parallel to the third side, the two sides are divided proportionally.*

Taking the preceding fig. in (69), through A draw Abc parallel to DEF, cutting BE in b, and CF in c. Then ACc will represent any triangle having its two sides AC, Ac, intersected in B, b, by Bb which is parallel to Cc; and it is required to prove that AB is to BC as Ab is to bc.

Since $ADEb$ is a parallelogram, $Ab = DE$. Similarly $bc = EF$; and, by (69), it is proved that AB is to BC as DE is to EF, $\therefore$ AB is to BC as Ab is to bc; that is, AC, Ac are divided *proportionally* in B, b.

Cor. *It follows that AB is to AC as Ab is to Ac.*

For since AB is to BC as Ab is to bc, this means that AB contains, or is contained in, BC, the same number of times that Ab contains, or is contained in, bc. Now it is plain that, whatever be the number of times AB contains, or is contained in, BC, it is contained in, $AB+BC$, or AC, exactly once more. Also whatever be the number of times Ab contains, or is contained in, bc, it is contained in, $Ab+bc$, or Ac, exactly once more. But, if each of two *equal* numbers be increased by 1, they will remain equal; $\therefore$ AB is contained in AC the same number of times that Ab is in Ac; that is, the *ratio* of AB to AC is *equal* to the *ratio* of Ab to Ac.

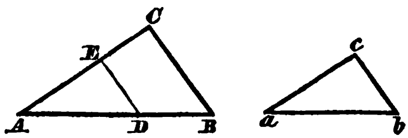
The common mode of writing the two last results is,

$$AB : BC :: Ab : bc, \text{ and } AB : AC :: Ab : Ac.$$

71. Prop. V. *Similar triangles have the sides forming the equal angles proportionals.*

[Def. *Similar triangles* are such as have their *angles equal, each to each.*]

Let ABC, abc, be *similar* triangles, that is, $\angle A = \angle a$,



$\angle B = \angle b$, and $\angle C = \angle c$; it is required to shew that

$$ab : AB :: ac : AC,$$
$$ab : AB :: bc : BC,$$
$$\text{and } bc : BC :: ac : AC.$$

With centre A and radius ab describe a circle cutting AB in D, and another circle with the same centre and radius ac cutting AC in E, and join ED. Then in the two triangles ADE, abc, the two sides AD, AE, are equal to the two sides ab, ac, each to each, and $\angle DAE + \angle bac$, $\therefore$ the triangles are equal in all respects (24), and $\therefore \angle ADE = \angle abc$. But $\angle abc = \angle ABC$, $\therefore \angle ADE = \angle ABC$, and $\therefore DE$ is *parallel* to BC (34). Hence by (70 Cor.) $AD : AB :: AE : AC$; but $AD = ab$, and $AE = ac$,

$$\therefore ab : AB :: ac : AC.$$

In the same way, by making B the centre of the circles, it may be shewn, that

$$ab : AB :: bc : BC;$$

and by making C the centre, that

$$bc : BC :: ac : AC.$$

COR. 1. Conversely, if two triangles have an angle of one equal to an angle of the other, and the sides forming the equal angles *proportionals*, the triangles will be *similar*.

COR. 2. Hence, also, if a triangle be cut off from a larger triangle by a line parallel to one of the sides, the two triangles will be *similar*, and have the sides about equal angles *proportionals*.

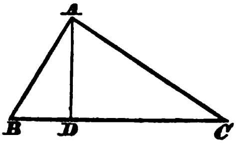
COR. 3. If CD be drawn perpendicular to AB, and cd to ab, it will also easily appear, that

$$CD : cd :: AC : ac, \text{ or } :: AB : ab, \text{ or } :: BC : bc.$$

72. PROP. VI. *If a right-angled triangle be divided into two other right-angled triangles by a straight line drawn from the vertex of the right angle perpendicular to the opposite side, each of these two triangles shall be* similar *to the whole triangle and to one another.*

Let ABC be a right-angled triangle having $\angle BAC$ the right angle. From A draw AD perpendicular to BC; then the triangles ABD, ADC shall be *similar* to the triangle ABC, and to each other.

Because $\angle BAC$ = a right angle $= \angle ADB$, and $\angle B$ is common to the two triangles ADB, ABC; and since the three angles of every triangle are together equal to two right angles, (37), $\therefore$ the remaining $\angle BAD$ of the one triangle is equal to the remaining $\angle ACB$ of the other, that is, the triangles ADB, ABC, are equiangular, and $\therefore$ *similar*.



In the same way it may be shewn, that the triangles ADC, ABC, are equiangular, and $\therefore$ *similar*. Hence also the triangles ADB, ADC are equiangular and *similar*.

COR. 1. Since in similar triangles the sides forming equal angles are *proportionals* (71); and since the triangles ADB, ADC are *similar*, $\therefore BD : AD :: AD : DC$.

Again, since ABD and ABC are *similar* triangles,

$$BD : AB :: AB : BC.$$

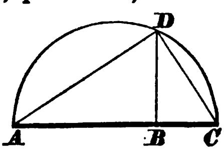
Also, since ADC and ABC are similar,

$$CD : AC :: AC : BC.$$

[DEF. When of four magnitudes which are proportionals the second and third are the same, this latter magnitude is said to be a *mean proportional* between the other two.

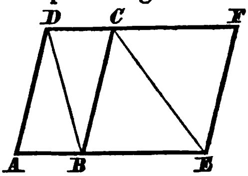
Thus, in this Cor. AD is a *mean proportional* between BD and DC. Also AB is a *mean proportional* between BD and BC; and AC is a *mean proportional* between BC and CD.]

COR. 2. Hence *to find a mean proportional between two given straight lines*, AB, BC, place AB, BC so as to form one straight line AC. Upon AC describe a semicircle; through B draw BD at right angles to AC meeting the circumference in D; and join AD, CD. Then, since $\angle ADC$ is a right angle, and DB is perpendicular to AC, by Cor. 1, $AB : BD :: BD : BC$; $\therefore BD$ is a *mean proportional* between AB and BC.



73. PROP. VII. *The areas of parallelograms or triangles between the same parallels are proportional to their bases.*

(1) Let $ABCD$, $BEFC$ be two parallelograms between the same parallels ABE, DCF, and upon the *bases* AB, BE, respectively.



Let such a lineal unit of measurement be taken as will exactly divide both AB, and BE; and divide AB and BE into as many *equal* parts as they contain the unit (68). Through the several points of division draw lines parallel to AD or BC, dividing $ABCD$ into as many parallelograms as the unit is contained in AB, and $BEFC$ into as many as the unit is contained in BE. Then since parallelograms upon equal bases and between the same parallels are equal to one another (41 Cor. 1), the smaller parallelograms which make up $ABCD$, and $BEFC$, are all equal. Therefore the ratio of $ABCD$ to $BEFC$ will be the ratio of the sum of these equal parallelograms in the one to the sum of them in the other, that is, as the *number* of them in the one to the *number* in the other, (since they are all equal) or as the number of units in AB is to the number of units in BE, that is, as AB is to BE.

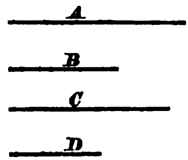
(2) Again, since a parallelogram is *double* of the triangle upon the same base and between the same parallels (40), and the *halves* of two magnitudes will plainly bear the same ratio to each other that the *whole* magnitudes do, ∴ joining BD, EC, the triangle ABD, which is half of the parallelogram $ABCD$, will have the same ratio to the triangle BEC, which is half of $BEFC$, that AB has to BE.

COR. It follows also, that the areas of *triangles* or *parallelograms* of *equal altitudes, however situated*, are proportional to their *bases;* the *altitude* being the perpendicular let fall from the vertex of one of the angles upon the opposite *side* considered as the *base*.

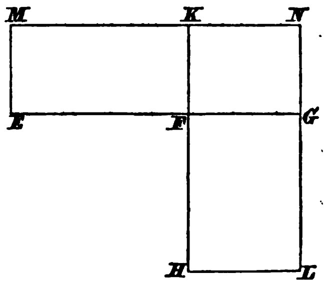
74. PROP. VIII. *If four straight lines taken in order be 'proportionals', the rectangle contained by the*

*first and fourth is equal to the rectangle contained by the second and third.**

Let A, B, C, D be four straight lines, '*proportionals*', that is, A is to B as C to D. The rectangle contained by A and D shall be equal to the rectangle contained by B and C.



Draw the straight line EF equal to A; produce it to G making FG equal to B; through F draw FH, FK at right angles to EF, making $FH = C$, and $FK = D$; through E and G draw EM, LGN parallel to HK; and through H, and K draw HL, and MKN parallel to EFG. Then $EFKM$, $FGNK$, and $FHLG$ are all rectangular parallelograms, as will easily appear. Now since $EFKM$ and $FGNK$ are parallelograms between the same parallels,



$EFKM : FGNK :: EF : FG$ (73), that is $:: A : B$.

But $A : B = C : D$, since A, B, C, D are proportionals,

$$\therefore EFKM : FGNK :: C : D.$$

Again, $FHLG : FGNK :: FH : FK$, that is $:: C : D$,

$$\therefore EFKM : FGNK :: FHLG : FGNK,$$

which signifies that $EFKM$ is the same multiple, part, or parts of $FGNK$ that $FHLG$ is of the same magnitude $FGNK$, $\therefore$ it is plain that $EFKM = FHLG$. But $EFKM$ is the rectangle contained by EF, FK, that is, A and D; also $FHLG$ is the rectangle contained by FG, FH, that is, B and C; $\therefore$ the rectangle contained by A and D is equal to the rectangle contained by B and C.

* This is sometimes expressed by saying, 'if four straight lines be proportionals, the rectangle contained by the *extremes* is equal to the rectangle contained by the *means*'.

Cor. 1. Conversely if A, B, C, D be any four straight lines, such that the rectangle contained by A and D is equal to the rectangle contained by B and C, then A, B, C, D are *proportionals*.

Cor. 2. If A, B, C, D be proportionals, since the rectangle $B, C =$ the rectangle A, D, it follows, from Cor. 1, that B, A, D, C are proportionals, that is,

$$B : A :: D : C.$$

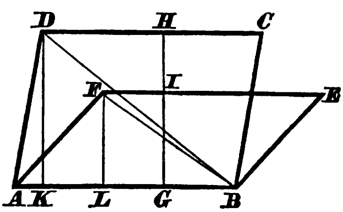
This change of *position* in the different members of a proportion is called '*invertendo*', or '*by inversion*'.

Cor. 3. Since the rectangle contained by C and B is equal to the rectangle contained by B and C, if A, B, C, D are proportionals, it follows that the rectangle $A, D =$ the rectangle C, B, and $\therefore A, C, B, D$ are proportionals, that is, $A : C :: B : D$. This is called '*alternando*', or '*alternately*'.

Cor. 4. Since the measure of the ratio of one *area* to another is simply the *number of times* the one contains or is contained in the other, this ratio may always be represented by the ratio of one *line* to another. Hence the two preceding Corollaries hold also for *areas* as well as *lines*, that is, if A and B be two *areas*, and C and D two *lines*, or if A, B, C, D be four *areas*, such that $A : B :: C : D$, then *inversely* $B : A :: D : C$. Also if A, B, C, D be *four areas* proportionals, such that $A : B :: C : D$, then, *alternately*, $A : C :: B : D$.

75. Prop. IX. *The areas of parallelograms, or triangles, on the same base, are proportional to their 'altitudes'.*

(1) Let $ABCD$, $ABEF$ be two parallelograms, on the same base AB; from any point G in AB draw GIH at right angles to AB, meeting CD in H, and EF in I. Then GH is the *altitude* of the parallelogram $ABCD$, and GI the *altitude* of $ABEF$.



And $ABCD$ shall be to $ABEF$ as GH is to GI.

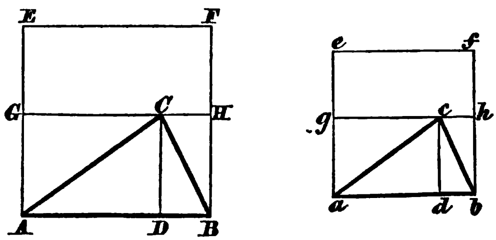
Let such a lineal unit of measurement be taken as will exactly divide both GH and GI; and divide GH and GI into as many equal parts as they contain this unit. Through the several points of division draw lines parallel to AB, dividing $ABCD$ into as many parallelograms as the unit is contained in GH, and $ABEF$ into as many as the unit is contained in GI. These smaller parallelograms are obviously all equal to one another; and therefore the area $ABCD$ will be to the area $ABEF$ in the same ratio as the number of them in the former area is to the number in the latter, that is, as the number of units in GH is to the number in GI, that is, as GH is to GI.

(2) Again, join BD, BF; then ABD, ABF will represent any two triangles on the same base AB. From D and F draw DK, FL perpendiculars to AB; then DK, FL are the '*altitudes*' of the triangles ABD, ABF. Also $DK = GH$, and $FL = GI$ (35 Cor.) Now ABD is half of the parallelogram $ABCD$; and ABF is half of $ABEF$; and the *halves* of two magnitudes must obviously have the same ratio to one another which the whole magnitudes have; $\therefore$ the triangle ABD : the triangle ABF :: GH : GI, that is, :: DK : FL; or the triangles are proportional to their '*altitudes*'.

COR. It follows also that parallelograms, or triangles, upon *equal* bases are proportional to their *altitudes*.

76. PROP. X. *The areas of* similar *triangles are proportional to the squares of any two corresponding sides**, *that is, sides opposite to equal angles*†.

Let ABC, abc be similar triangles, in which $\angle A = \angle a$,



* Sometimes called '*homologous* sides'.

† Euclid's enunciation of this is: '*Similar triangles are to one another in the duplicate ratio of their homologous sides*'.

$\angle B = \angle b$, $\angle C = \angle c$; then AB, ab being *any* two corresponding, or homologous, sides, the triangle ABC shall be to the triangle abc as the square of AB is to the square of ab.

Upon AB describe the square $AEFB$, and upon ab the square $aefb$. From C draw CD perpendicular to AB, and from c draw cd perpendicular to ab. Through C draw GCH parallel to AB, and through c draw gch parallel to ab. Then, since a triangle is always equal to half the parallelogram upon the same base and between the same parallels,

triangle ABC : triangle abc :: paral^m. $AGHB$: paral^m. $aghb$.

Now paral^m. $AGHB$: square of AB :: AG : AE,

i. e. :: CD : AB, (73),

and paral^m. $aghb$: square of ab :: ag : ae,

i. e. :: cd : ab;

but $CD : AB = cd : ab$ (71, Cor. 3, and 74, Cor. 3), since ABC, abc, are *similar* triangles,

$\therefore$ paral^m. $AGHB$: square of AB :: paral^m. $aghb$: square of ab;

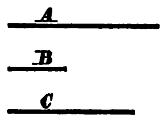
and, *alternately*,

paral^m. $AGHB$: paral^m. $aghb$ :: square of AB : square of ab,

$\therefore$ triangle ABC : triangle abc :: square of AB : square of ab.

[This is one of the most important Theorems in Geometry.]

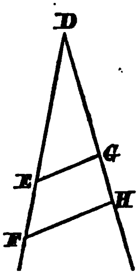
77. Prop. XI. *To find a fourth proportional to three given straight lines, that is, a fourth line such that the four lines shall be proportionals.*



Let A, B, C be the three given straight lines; it is required to find another X, such that $A : B :: C : X$.

Draw any indefinite straight line DEF, in which take DE equal to A, and EF equal to B. From D draw DGH making any angle with DF, in which take DG equal to C. Join EG, and through F draw FH parallel to EG, meeting the line DGH in H.

Then since DFH is a triangle, and EG is parallel to FH, $DE : EF :: DG : GH$ (70). But $DE = A$, $EF = B$, $DG = C$; $\therefore A : B :: C : GH$, that is, $GH = X$, the straight line required.



COR. By the same method a *third proportional* may be found to two given straight lines, that is, a third line C such, that $A : B :: B : C$. The only difference is, that DG is taken equal to EF and equal to B. Then

$$DE : EF :: DG : GH,$$

that is, $A : B :: B : GH$, $\therefore GH = C$.

78. DEF. *Four-sided* figures are *similar*, when they have their angles equal, *each to each*, and the sides forming equal angles *proportionals.*

Hence *all squares* are similar figures, since the angles of any one are equal to the angles of any other, each to each, and the sides about equal angles (being equal) are *proportionals.*

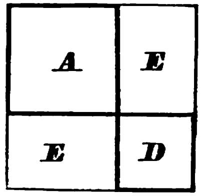
But neither two *rectangles*, nor two *parallelograms* with angles equal each to each, are necessarily *similar*. In addition to the equality of angles, the sides about the equal angles must be *proportionals;* and although, in the case of *triangles*, it follows as a consequence of the equality of *angles*, that the *sides* about equal angles *are* proportionals (71), yet it is not so with any other rectilinear figures, except squares, as may easily be shewn. For instance, if a part be cut off from a parallelogram by a straight line parallel to one of the sides, the new parallelogram will have its *angles* equal to those of the original one, each to each; but it is obvious that the *sides* about equal angles in each are not proportionals; and therefore the parallelograms are not *similar.*

79. PROP. XII. *If the squares described upon four straight lines be proportionals, the straight lines themselves are proportionals; and conversely.*

Let A, B, C, D* represent four squares, *proportionals*, that is, $A : B :: C : D$.

* In this proposition a single letter is used to designate a square or a rectangle, contrary to rule, merely to avoid unnecessary writing. This

Let A, and D, be so placed that two sides of one square may be in the same straight lines with two sides of the other, each to each; and produce the other sides of A and D until they meet. The resulting fig. will be a *square*, composed of the two squares A and D, together with two equal rectangles, E, E. Then since A and E are parallelograms of the same altitude,

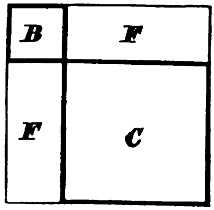


$$A : E :: \text{base of } A : \text{base of } E.$$

$$\text{Also } E : D :: \text{base of } A : \text{base of } E;$$

$$\therefore A : E :: E : D.$$

The same construction being made with B and C, as in the annexed fig., it may be shewn in the same manner that $B : F :: F : C$.



Now let a, b, c, d, e, f represent straight lines having the same ratio to each other as the *areas* A, B, C, D, E, F. Then, since $A : B :: a : b$, $C : D :: c : d$, and $A : B :: C : D$, $\therefore a : b :: c : d$; and therefore the rectangle contained by a and d = the rectangle contained by b and c. Similarly, $a : e :: e : d$, and $b : f :: f : c$; $\therefore$ rectangle a, d = square of e, and rectangle b, c = square of f; $\therefore$ square of e = square of f, and $\therefore e = f$ (42 Cor. 3). But $E : F :: e : f$, $\therefore E = F$.

Now E is the rectangle contained by a side of A and a side of D; and F is the rectangle contained by a side of B and a side of C; and rectangle E = rectangle F; therefore (74 Cor. 1),

side of A : side of B :: side of C : side of D.

Cor. 1. Hence also the converse is easily shewn, viz. that, if four straight lines are *proportionals*, the squares described upon them are *proportionals*.

Cor. 2. In (76) it was proved that the areas of similar triangles are to one another as the squares of corre-

may be done here without inconvenience, because we are not much concerned with the magnitude of the *sides* until we arrive at the last stage proof.

sponding, or '*homologous*', sides. It may now be shewn that the areas are proportional to the squares of the *altitudes* also of the triangles, or to the squares of *any* corresponding lines within the triangles. For (see fig. in 76) $CD : cd = BC : bc = AB : ab$, $\therefore$ square of CD : square of cd = square of AB : square of ab; and $\therefore$ triangle ABC : triangle abc :: square of CD : square cd.

80. Prop. XIII. *If there be any number of magnitudes, which, taken two and two, have a certain fixed ratio to each other, the* sum *of the first terms of the several pairs of magnitudes shall be to the sum of their second terms in that same ratio.*

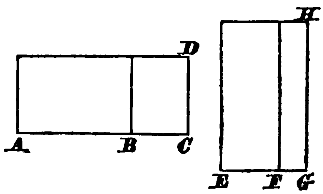
Let A, B, C, D be four magnitudes such that $A : B :: E : F$, and $C : D :: E : F$, then also

$$A + C : B + D :: E : F.$$

For suppose a, b, c, d, e, f to represent six straight lines having the same ratio to each other that A, B, C, D, E, F, have. Then since $a : b :: e : f$, the rectangle a, f = the rectangle b, e (74). Similarly, the rectangle c, f = the rectangle d, e.

$$\therefore \text{rectangle } a, f + \text{rectangle } c, f = \text{rectangle } b, e$$
$$+ \text{rectangle } d, e.$$

Now, constructing each of these rectangles, as in the annexed figs. by making $AB = a$, $BC = c$, $CD = f$; also $EF = b$, $FG = d$, $GH = e$; it is obvious that the rectangle a, f + rectangle c, f = rectangle contained by $AB + BC$, and CD, that is, rectangle $a + c, f$.



Also rectangle b, e + rectangle d, e = rectangle contained by $EF + FG$, and GH, that is, rectangle $b + d, e$.

$$\therefore \text{rectangle } a + c, f = \text{rectangle } b + d, e,$$

$$\text{and } \therefore a + c : b + d :: e : f \text{ (74 Cor. 1).}$$

$$\text{But } a + c : b + d :: A + B : B + D,$$

$$\text{and } e : f :: E : F, \text{ by supposition,}$$

$$\therefore A + C : B + D :: E : F.$$

If there be another proportion $G : H :: E : F$, then *it follows*, from what has been proved, considering $A + C$, as a single magnitude, and likewise $B + D$, that

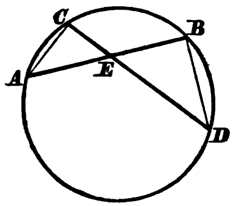
$$A + C + G : B + D + H :: E : F;$$

and so on, whatever be the number of proportions having the same two last terms in each.

Cor. Hence, since two similar *parallelograms* are composed of two pairs of similar triangles, which are to each other as the squares of corresponding sides or lines within them (76), therefore the *parallelograms* also are proportional to the squares of their homologous sides or other corresponding lines within them.

81. Prop. XIV. *If any two chords be drawn in the same circle intersecting each other, the rectangles contained by the parts into which each is divided by the point of intersection are equal to one another.*

Let AB, CD be two chords of a given circle intersecting in E. Then the rectangle AE, EB shall be equal to the rectangle CE, ED.



Join AC, BD. Then $\angle ACD = \angle ABD$, being angles in the same segment (52 Cor.), that is, $\angle ACE = \angle EBD$. Similarly, $\angle CAE = \angle BDE$. Also $\angle AEC = \angle BED$ (31); $\therefore$ the triangles AEC, BED, are *similar;* and $\therefore$ the sides about equal angles are proportionals; that is,

$$AE : EC :: ED : EB \text{ (71)},$$

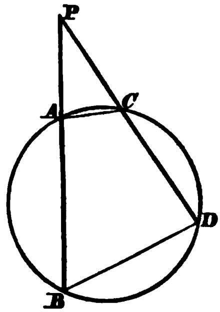
and $\therefore$ the rectangle AE, EB = the rectangle CE, ED (74).

Cor. 1. If one of the chords, AB, *bisect* the other CD, in the point E, then it follows that the rectangle AE, EB is equal to the *square* of CE.

Cor. 2. If AB bisect CD *at right angles*, AB will be a *diameter*, and then also the rectangle AE, EB = the square of CE.

82. Prop. XV. *If any two chords of the same circle be produced to meet in a point without the circle, the rectangles contained by the whole line and the part produced, for each chord, shall be equal to one another.*

Let AB, CD be two chords of the same circle, which produced meet in the point P; the rectangle PA, PB shall be equal to the rectangle PC, PD.

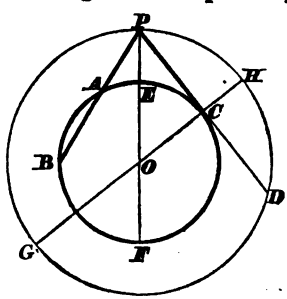


Join AC, BD; then since $ABDC$ is a quadrilateral 'inscribed' in a circle, $\angle ABD + \angle ACD =$ two right angles (53), $= \angle ACP + \angle ACD$ (30), $\therefore \angle ABD = \angle ACP$. And angle at P is common to the two triangles PBD, PAC; $\therefore$ remaining angle $PAC =$ remaining angle PDB; and $\therefore$ the triangles PBD, PAC are *similar*, and the sides about equal angles proportionals (71), $\therefore PA : PC :: PD : PB$, and $\therefore$ the rectangle PA, $PB =$ the rectangle PC, PD (74).

Cor. If one of the chords, AB, be a *diameter*, the proposition holds true, viz. that the rectangle PA, PB = the rectangle PC, PD.

83. Prop. XVI. *If any chord and tangent of the same circle be produced until they meet, the rectangle contained by the whole chord thus produced and the part produced shall be equal to the square of the tangent, that is, of the line between its point of intersection with the chord produced and the point where it touches the circle.*

Let AB be a chord, and PC a tangent at the point C, of the circle ABC, whose centre is O; and let BA produced meet PC in the point P. Then the rectangle PA, PB shall be equal to the square of PC.



Join PO, and produce it to the circumference of the circle, so that it meets the circumference in E and F, making EF a *diameter*. With centre O and radius OP describe another circle, that is, concentric with the former;

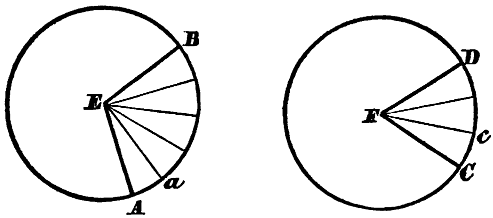
join OC, and produce it both ways to meet the outer circumference in G and H, so that GH is a *diameter;* and produce PC to meet this circumference in D, so that PD is a *chord* of the outer circle.

Then, since GH is a *diameter* at right angles (55 Cor. 1) to the *chord* PD, PD is *bisected* in C (49 Cor.); and $\therefore$ (81 Cor. 2) the rectangle GC, $CH =$ the square of PC. But $GC = PF$, and $CH = PE$; $\therefore$ the rectangle PE, PF = the square of PC. But the rectangle PE, $PF =$ the rectangle PA, PB (82),

$\therefore$ the rectangle PA, $PB =$ the square of PC.

84. Prop. XVII. *In the same circle, or in equal circles, any two arcs are proportional to the angles which they subtend at the centre.*

Let AB, CD be any arcs of two equal circles; find the centres E, F, and join EA, EB, FC, FD. Then arc AB : arc CD :: $\angle AEB : \angle CFD$.



For, assuming that there is some small arc which, taken as the unit of measurement, is contained an exact number of times both in AB and CD, let Aa be such arc, and suppose it to be contained 5 times in AB, and 3 times in CD, (the proof is the same whatever the numbers be); draw the radii, as Ea, and Fc, to the several points of division in AB and CD, so that the arc AB is divided into 5 equal parts, CD into 3 equal parts, and also the angles AED, CFD, into 5 and 3 equal *angles*, respectively, since equal arcs in the same circle, or in equal circles, subtend equal angles at the centre (59). Then, since arc AB contains a certain unit 5 times, and arc CD the same unit 3 times, arc AB : arc CD :: 5 : 3. Again $\angle AED$ contains $\angle AEa$ 5 times, and $\angle CFD$ contains $\angle CFc$, which is equal to $\angle AEa$, 3 times,

$\therefore \angle AEB : \angle CFD :: 5 : 3.$

And $\therefore$ arc AB : arc CD :: $\angle AEB : \angle CFD$.

NOTE.

It will have been noticed, that, in the preceding Theory of *Ratio* and *Proportion*, the magnitudes compared are assumed to be, what is called, '*commensurable*', that is, to have a '*common measure*', or common *unit* of measurement. Now two or more magnitudes are said to have a '*common measure*', when each of them contains the *unit* of measurement a certain number of times exactly *without remainder*.

Thus two lines, which are $5\frac{1}{3}$ yards and $7\frac{2}{3}$ yards in length respectively, are *commensurable*, because, taking the *foot* as the *measure*, the first line contains it 16 times, and the second 23 times, *exactly*. Similarly, two lines which are $2\frac{1}{2}$ yards and $1\frac{1}{2}$ yards in length respectively, are *commensurable*, because the first contains an *inch* 90 times, and the second 54 times, *exactly*.

But it does not follow, (and in fact it is not true,) that *all* lines are of this kind, that is, *commensurable*. *Lines*, and also *areas*, have sometimes to be compared, which have no *common measure*, and are called *incommensurable*. To these the preceding Theory does not with *perfect mathematical accuracy* apply, as it does to *commensurable* magnitudes; although in all such cases a measure may be found which shall approach *as nearly as we please* to a *common* measure, and thus render the preceding Theory applicable *by approximation*, and to all *practical* purposes *sufficiently* true.

Euclid's method of treating *ratios* and *proportion*, which applies strictly and equally to all magnitudes, *commensurable* and *incommensurable*, has not been adopted, simply because it does not admit of being presented in a form sufficiently intelligible to those for whom this little work is designed. It seemed better to employ a method, which, with admitted imperfections, would allure the learner, than to aim at a perfectness of theory, which might lead him either to pass over the subject entirely, or to read it and not understand it.

EXERCISES C.

(1) Define '*ratio*'; between what sort of magnitudes can it exist? Is there any '*ratio*' between ten shillings and two miles? If not, why not?

(2) What is the *test* by which you determine whether, or not, two proposed magnitudes are '*of the same kind*'? Apply it to the case of a triangle and one of its sides. Also to the case of a triangle and a square.

(3) Is there any *ratio* between an *angle* and a *triangle*? Or between a *right angle* and a *square*?

(4) Define the *measure* of a *ratio*; and express the *ratio* of a *parallelogram* to the *triangle* 'on the same base and between the same parallels'.

(5) What is the *ratio* of a lineal *inch* to a lineal yard?

(6) What is the *ratio* of the *square* of AB to the *square* of the half of AB?

(7) Is it necessary, when two lines or magnitudes have a *ratio* to each other, that the one should contain the other an exact integral number of times? Explain fully.

(8) When is one line, or area, said to be a *multiple* of another? If 5 *times* $A = 7$ *times* B, what is the *ratio* of A to B?

(9) Define '*proportion*', and '*proportional*'. How many magnitudes are concerned in a *proportion*? *May* they be all *of one kind*? *Must* they be so?

(10) Can two *lines* and two *triangles* be in *proportion*? Can two *angles*, a *triangle*, and a *parallelogram*, be in *proportion*?

(11) If there be two triangles of equal altitudes, and the base of one be double the base of the other, what is the *proportion* between the bases and triangles? And what is the *ratio* of the two triangles?

(12) Shew that, if any two sides of a triangle be bisected, the line joining the points of bisection is *parallel* to the third side, and equal to half of it.

(13) Define *similar* triangles; can triangles be *similar* and not *equal?* Can they be *equal* and not *similar?* Explain fully.

(14) Are all *equilateral* triangles *similar?* Are two *isosceles* triangles necessarily *similar?*

(15) If each of the sides of a triangle be bisected, shew that the lines joining the points of bisection will divide the triangle into four equal triangles *similar* to the whole triangle and to each other.

(16) If through the vertex of each angle of a triangle a straight line be drawn parallel to the opposite side, shew that these lines will form a triangle *similar* to the given triangle; and find the *ratio* of this triangle to the given triangle.

(17) If the sides of *any* quadrilateral figure be bisected, shew that the lines joining the points of bisection will form a *parallelogram.*

(18) Shew that any triangle cut off from an equilateral triangle by a line parallel to one of its sides is equilateral.

(19) Through a given point draw a straight line, terminated by two given straight lines, so that it shall be *bisected* in that point.

(20) Through a given point draw a straight line, terminated by two other given straight lines, so that it shall be divided by that point *in a given ratio.*

(21) Of all triangles with two given sides shew that that is the greatest in which the two sides form a *right angle.*

(22) If an angle of a triangle be bisected by a straight line which also cuts the opposite side, shew that the two parts into which this side is divided will be in the same ratio as the other two sides are to one another.

(23) Shew that any two *right-angled* triangles are *similar,* if two of their acute angles, one in each triangle, are equal.

(24) If two triangles have the sides of the one, or sides produced, respectively at right angles to those of

the other, each to each, shew that the triangles are *similar.*

(25) If each of the sides of a triangle be bisected, and straight lines be drawn from the points of bisection to the vertex of the opposite angle, shew that these three lines will intersect in one point, and that the point of intersection divides each line into two parts of which one is double the other.

(26) In the last problem shew that the three lines from the point of intersection to the vertices of the three angles divide the given triangle into three *equal* triangles.

(27) Shew that two isosceles triangles will be *similar,* if *any* angle of the one be equal to the corresponding angle of the other.

(28) Find the *greatest* 'mean proportional' between any two lines of given *sum.*

(29) If two circles touch each other, either internally or externally, and two straight lines be drawn through the point of contact; so as to form four chords, two in each circle, shew that the four chords are *proportionals.*

(30) If two circles touch each other externally, and a straight line be drawn touching both and terminating at the points of contact, shew that this line is a *mean proportional* between the diameters.

(31) Shew that the parts into which the diagonals of a trapezium are divided by their point of intersection are *proportionals.*

(32) Shew that any rectangle is a mean proportional between the squares of two of its adjacent sides.

(33) Shew *geometrically* that a side of a *square* and its diagonal are '*incommensurable*'.

(34) If on the sides of a right-angled triangle, taken as *bases,* three *similar* rectangles be described, shew that the rectangle on the side opposite to the right angle is equal to the sum of the other two.

POLYGONS, AND THEIR CONNECTION WITH THE CIRCLE.

85. Definition. A Polygon* is a plane surface bounded by more than four straight lines, which are called its *sides*.

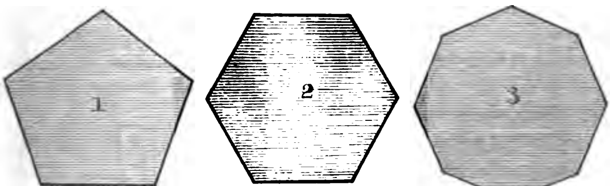
A plane surface with three *sides* has already received the name of *triangle*, and with *four* sides '*parallelogram*', '*square*', '*quadrilateral*', or '*trapezium*', as the case may be; therefore *polygons* begin with *five* sides, and may have any greater number.

An *angle of a* polygon means an angle formed by two *adjacent sides* of the polygon. And the number of the *angles* is obviously equal to the number of the *sides*.

Def. A *Polygon* of 5 sides is called a *Pentagon*†,
.... 6 .... *Hexagon*‡,
.... 8 .... *Octagon*‖;
and so on.

Def. A *Regular Polygon* is a polygon which has all its angles equal and all its sides equal.

Thus a regular Pentagon, Hexagon, and Octagon will respectively present the following appearance as to form:



[It does not *yet* appear that a *regular polygon*, as here defined, is a *possible* construction. All that is meant is, that, if such be possible, these are the distinctive names of such polygons §.]

Def. The *sum* of all the *sides* of a polygon is called its *perimeter*.

* *Polygon*, derived from two Greek words, literally means a figure which has many corners.

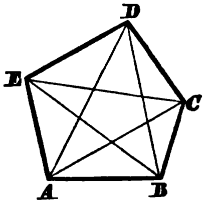
† *Pentagon*, that is, a *five-cornered* figure.

‡ *Hexagon*, that is, a *six-cornered* figure.

‖ *Octagon*, that is, an *eight-cornered* figure.

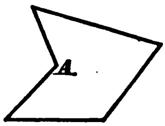
§ A similar observation might have been made, when the Definitions of *equilateral* triangle, and of a *square*, were given. We were not then able to say, that such constructions were *possible*.

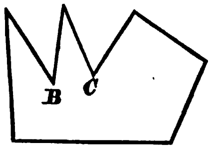
DEF. A straight line, drawn from the vertex of any angle to the vertex of any other angle not adjacent to the former, is called a *diagonal*.



Thus, in the annexed fig. $AB + BC + CD + DE + EA$ is the *perimeter*, and each of the straight lines AC, AD, BE, BD, EC, is a *diagonal*, of the *polygon* $ABCDE$.

N. B. Throughout this section all those *polygons* are excluded which have what are called '*re-entrant angles*', such as the polygons annexed:

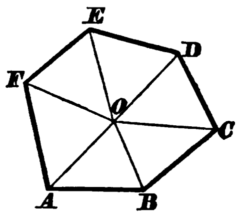




where A, B, C are *re-entrant* angles. They are called *re-entrant* angles, because if the lines forming them be produced through the vertex, these lines *enter within* the polygon, which is not the case with *ordinary* polygons.

86. PROP. I. *All the angles of a polygon are together equal to twice as many right angles as the polygon has sides, diminished by four right angles.*

For every polygon, as $ABCDEF$, may be divided into triangles by taking any point O *within* the polygon, and joining OA, OB, OC, OD, OE, OF; and the number of triangles will obviously be the same as the number of the sides of the polygon. But the three angles of each triangle are together equal to two right angles; ∴ the angles of all the triangles are together equal to twice as many right angles as the polygon has sides; that is, all the angles of the polygon, together with the angles having the common vertex O, are equal to twice as many right angles as the polygon has sides. But the angles at O are



equal to four right angles (30 Cor.); ∴ all the angles of the polygon are equal to twice as many right angles as the polygon has sides, diminished by four right angles.

Cor. 1. Hence, all the angles of a pentagon = 6 right angles;
.... hexagon = 8 ....;
.... octagon = 12 ....;

and so on, whatever be the number of sides of the polygon*.

Hence, also, since all the angles are equal to one another in a *regular* polygon,

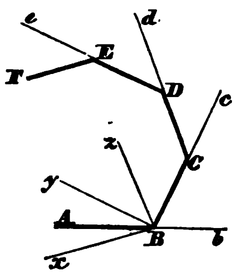
each angle of a regular pentagon = $\frac{6}{5}$ of a right angle;

.... hexagon = $\frac{4}{3}$ ....;

.... octagon = $\frac{3}{2}$ ....;

and so on.

Cor. 2. If $ABCDEF$ be a portion of the perimeter of any polygon; and if the sides AB, BC, CD, &c., be produced to b, c, d, &c., since each interior angle, as $\angle ABC$, + its exterior angle, as $\angle bBC$, = two right angles, ∴ all the interior angles + all the exterior angles = twice as many right angles as the polygon has sides; and ∴, by what has been proved, all the *exterior* angles of a polygon are together equal to four right angles.



The same result may also be made to appear from a very simple consideration. From B draw Bz parallel to CD, By parallel to DE, Bx parallel to EF, &c., taking every side of the polygon in succession. Then $\angle DCc = \angle CBz$, $\angle EDd = \angle yBz$, $\angle FEe = \angle xBy$, &c.; and the last of the lines Bz, By, Bx, &c., will be Bb; ∴ the sum of

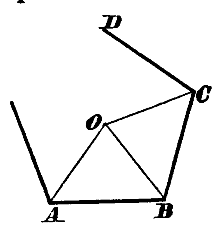
* The triangle, and quadrilateral, as we might expect, both follow the same rule. Thus all the angles of a triangle are equal to 6 right angles diminished by 4 right angles, that is, are equal to 2 right angles. And all the angles of a quadrilateral are equal to 8 right angles diminished by 4 right angles, that is, are equal to 4 right angles.

the exterior angles will be equal to the sum of the angles occupying the whole space round B as a common vertex, that is, four right angles (30 Cor.).

Cor. 3. Since the magnitude of each angle in a *regular* polygon depends only on the *number*, and not the *length*, of the sides, therefore all regular polygons *of the same name* have precisely the same angles, however much they may differ in their other dimensions. Hence all *regular* polygons *of the same name* are *similar*, in the sense in which certain *triangles* were defined to be *similar*, for besides equal angles, each to each, such polygons having *equal* sides throughout each, will, of course, have the sides about equal angles *proportionals*.

87. Prop. II. *In every* regular *polygon if lines be drawn severally bisecting the angles, these lines will all meet in the same point within the polygon; and that point will be equidistant from all the angular points in the perimeter of the polygon.*

Let $ABCD$ be a portion of the perimeter of a *regular* polygon. Bisect the angles at A and B by the straight lines OA, OB, meeting in O; and *join* OC. Then, since $\angle OAB =$ half $\angle A$, and $\angle OBA =$ half $\angle B$, and the angles of the polygon at A and B are equal, $\therefore$ $\angle OAB = \angle OBA$, and $\therefore$ $OA = OB$. Again, since $AB = BC$, and BO is common to the two triangles OAB, OBC, and $\angle OBC = \angle OBA$, $\therefore$ $OC = OA$, and $\angle OCB = \angle OBA$ (24). But $\angle OBA =$ half $\angle B =$ half $\angle C$, $\therefore$ OC bisects $\angle C$. Hence $OA = OB = OC$, and OA, OB, OC, bisect the angles at A, B, C. The same may be proved in the same manner for all the remaining angles of the polygon.



Cor. 1. Hence, if with centre O and radius OA a circle be described, its circumference will pass through *all* the angular points in the perimeter of the polygon.

In this case the circle is said to be '*described about*' the polygon, or the polygon to be '*inscribed in*' the circle.

Cor. 2. Since AB and BC are given *chords* of this

circle, it is obvious also (50), that the centre O may easily be determined by bisecting AB, and BC, and through the points of bisection drawing lines at right angles to AB, BC, and meeting, as they will do, in O.

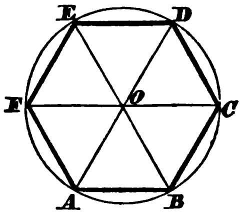
COR. 3. If a circle be described with centre O and radius equal to the perpendicular from O upon AB, every side of the polygon will be a *tangent* to this circle.

In this case the polygon is said to be '*described about*' the circle, or the circle to be '*inscribed in*' the polygon.

COR. 4. Hence *every regular* polygon may be inscribed in a given circle, or described about a given circle.

88. PROP. III. *In every* regular *polygon of an* even *number of sides to each side there is another opposite side* parallel *to it; and to each angle there is an opposite angle such that the* vertices *of the two are in the same diameter of the circumscribing circle.*

Let $ABCDEF$ be a *regular* polygon of *six* sides, (the proof will be the same for eight sides, ten sides, &c.) Find O the centre of the circumscribing circle (87), and join OA, OB, OC, OD, OE, OF. Then since *one half* of all the sides will be equal to the other half, and $AB = BC = CD =$ &c., and equal chords in the same circle cut off equal arcs (58), the three *arcs* AB, BC, CD are together equal to the three *arcs* DE, EF, FA, that is, $ABCD$ is a *semicircle*, and $\therefore AOD$ is a straight line and a *diameter*. Similarly BOE is a diameter, and FOC a diameter, of the circumscribing circle.



Again, since $OA = OB = OE = OD$, and $\angle AOB = \angle DOE$ (31), $\therefore$ the triangles AOB, DOE are equal in all respects, and $\angle OAB = \angle ODE$, $\therefore AB$ is parallel to ED (34). Similarly BC is parallel to EF; and CD to AF.

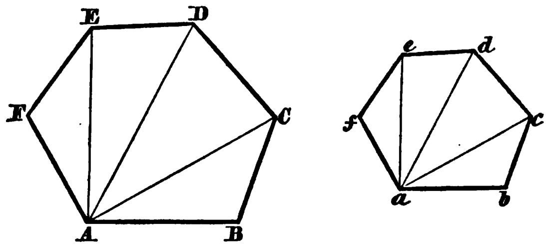
COR. 1. Hence, in the case of a hexagon, AOB is an *equilateral* triangle. For, since AOD is a straight line, and $\angle AOB = \angle BOC = \angle COD$, $\therefore \angle AOB =$ *one-third* of two right angles (30 Cor.), and $\therefore \angle OAB + \angle OBA =$

two-thirds of two right angles (37). But $\angle OAB = \angle OBA$, $\therefore$ *each* of them is *one-third* of two right angles; and $\therefore$ the triangle AOB is *equiangular;* and because it is equiangular it is also equilateral (26 Cor.).

Cor. 2. Hence, also, to construct a hexagon upon a given straight line AB, that is, having the given straight line for a *side,* it is only necessary to describe an equilateral triangle on the given line, as AOB; then produce AO, BO to D and E, making $OD = OA = OE$, which will determine the angular points D and E; then with centres B and D and radius OA describe two arcs intersecting in C, and with centres A and E and the same radius two arcs intersecting in F; join BC, CD, DE, EF, FA, and the required hexagon $ABCDEF$ is constructed.

89. Prop. IV. *Two* similar *polygons may be divided into the same number of* similar *triangles, each to each, and similarly situated.*

[Def. Two polygons are *similar,* when they have the same number of sides, and all the angles of the one are separately equal to all the angles of the other, *each to each,* and the sides also about equal angles proportionals.]



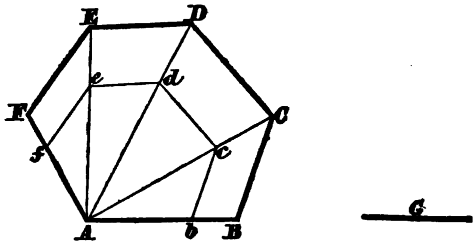
Let $ABCDEF$, $abcdef$, be two *similar* polygons, the angles at A, B, C, D, E, F, being equal to the angles at a, b, c, d, e, f, *each to each.* From A draw the diagonals AC, AD, AE; and from a draw the diagonals ac, ad, ae. Then since $\angle ABC = \angle abc$, and also $AB : ab :: BC : bc$, by Definition, $\therefore$ the triangles ABC, abc, are *similar* (71 Cor. 1), $\therefore$ also $\angle ACB = \angle acb$, and $AC : ac :: BC : bc$. But $\angle BCD = \angle bcd$, $\therefore$ $\angle ACD = \angle acd$; and $BC : bc :: CD : cd$,

by supposition, $\therefore$ $AC : ac :: CD : cd$. Hence again the triangle ACD is similar to the triangle acd. And in the same way it may be shewn that the triangles ADE, ade, are similar: and also that the remaining triangles AEF, aef are similar.

N. B. It is not enough in *polygons*, as in *triangles*, to make them *similar*, that the *angles* of the one are respectively equal to those of the other, because two *triangles cannot* have their angles respectively equal without having the sides about equal angles *proportional;* whereas this does not hold for *polygons*, seeing that we can alter the sides in an almost endless number of ways, without altering any angle. For instance, suppose we cut off a large part of the polygon $ABCDEF$ by a line *parallel* to BC and near to AD, the *angles* of the new polygon will be the same as those of $ABCDEF$, but it is obvious that the new polygon is not *similar* to $abcdef$, not having its *sides* in the same proportion.

Cor. The converse will easily follow, viz. that, if two polygons are composed of the same number of similar triangles, arranged in the same order in each polygon, the polygons shall be similar.

90. Prop. V. *Upon a given straight line to construct a polygon similar to a given polygon.*



Let $ABCDEF$ be the given polygon, and G the given straight line; it is required to construct upon G, that is, upon a *base* equal to G, a polygon similar to $ABCDEF$.

(1) Suppose G less than AB; with centre A and radius equal to G describe a circle cutting AB in b, making Ab equal to G; join AC, AD, AE; through b draw bc parallel to BC meeting AC in c; through c draw cd

parallel to CD meeting AD in d; through d draw de parallel to DE meeting AE in e; and through e draw ef parallel to EF meeting AF in f. Then $Abcdef$ shall be *similar* to $ABCDEF$, and it stands upon the base Ab equal to G.

For, since bc is parallel to BC, the triangles Abc, ABC are *similar*. So also Acd is *similar* to ACD; Ade to ADE; and Aef to AEF, $\therefore$ $\angle Abc = \angle ABC$; $\angle Acb = \angle ACB$; $\angle Acd = \angle ACD$, and $\therefore$ $\angle bcd = \angle BCD$. Similarly $\angle cde = \angle CDE$, $\angle def = \angle DEF$, and $\angle efA = EFA$. Hence $Abcdef$ and $ABCDEF$ are *equiangular*.

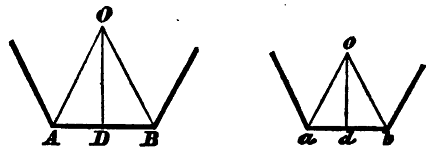
Again, by similarity of *triangles*, $AB : Ab :: BC : bc$; $AC : Ac :: CD : cd$, and $AC : Ac :: BC : bc$, $\therefore BC : bc :: CD : cd$. Similarly $CD : cd :: DE : de$; and $DE : de :: EF : ef$; and $EF : ef :: AF : Af$; $\therefore$ the sides about the equal angles are *proportionals*.

Hence $ABCDEF$ and $Abcdef$ are *similar* polygons.

(2) If G be greater than AB, *produce* AB, AC, AD, AE, AF indefinitely, and in AB produced take Ab equal to G, and proceed as before.

91. Prop. VI. *The* perimeters *of regular polygons of the same number of sides are proportional to the radii of their inscribed or circumscribing circles; and their* areas *are proportional to the squares of those radii.*

(1) Let AB, ab be sides of two regular polygons of the same name, that is, of the same number of sides;



O, o, the centres of their inscribed and circumscribing circles*. Join OA, OB, oa, ob; and draw OD perpendicular to AB, and od perpendicular to ab. Then $OA = OB$ = radius of circumscribing circle to one of the polygons, and $oa = ob$ = radius of circumscribing circle to the other polygon; OD = radius of inscribed circle to

* That the inscribed and circumscribing circles in the same regular polygon have the same centre appears from (80).

one of the polygons, od = radius of inscribed circle to the other polygon (84).

Again, since each side of a regular polygon subtends the same angle at the centre of the inscribed and circumscribing circle, $\angle AOB = \angle aob$, being angles which are the *same part* of 4 right angles.

Also, since $AO = BO$, and $ao = bo$, $\angle OAB = \angle OBA$, and $\angle oab = \angle oba$; but $\angle OAB + \angle OBA + \angle AOB$ = two right angles = $\angle oab + \angle oba + \angle aob$, $\therefore \angle OAB = \angle oab$, and $\angle OBA = \angle oba$, $\therefore$ OAB and oab are *similar* triangles. Hence $AB : ab :: OA : oa$, or $:: OD : od$; and every pair of sides is in the same ratio; therefore (80)

sum of the sides of one polygon : *sum* of the sides of the other $:: OA : oa$, or $:: OD : od$, that is, the *perimeters* of the polygons are as the radii of the inscribed or circumscribing circles.

(2) Again, since the polygons are made up of the same number of similar triangles, as AOB, aob; and since $AOB : aob$ :: square of AO : square of ao,

or :: square of OD : square of od,

$\therefore$ sum of these triangles in one polygon : sum of them in the other :: square of AO : square of ao,

or :: square of OD : square of od;

that is, the *areas* of the polygons are as the squares of the radii of the inscribed or circumscribing circles.

92. PROP. VII. *The areas of similar polygons are to one another as the squares of any homologous sides, or corresponding lines within the polygons.*

Let $ABCDEF$, $abcdef$ be two similar polygons, of which AB, ab are *any* two corresponding sides; then

area $ABCDEF : abcdef$ :: square of AB : square of ab.

From A, a, draw the diagonals AC, AD, AE, ac, ad, ae. These will divide the polygons into the same number of triangles, *similar* and *similarly situated*, each to each, see fig. (89).

$\therefore$ by (76),

triangle ABC : triangle abc :: square of AB : square of ab,

...... ACD : ...... acd :: square of CD : square of cd,

...... ADE : ...... ade :: square of DE : square of de,

...... AEF : ...... aef :: square of EF : square of ef,

But $AB : ab :: BC : bc :: CD : cd :: DE : de :: EF : ef$ (71),

$\therefore$ square of CD : square of cd :: square of AB : square of ab,

square of DE : square of de :: ....

square of EF : square of ef :: ....

$\therefore$ (80) $ABC + ACD + ADE + AEF : abc + acd + ade + aef$

:: square of AB : square of ab,

or area $ABCDEF : abcdef$ :: square of AB : square of ab.

Again, since $AB : ab :: AC : ac :: AD : ad :: AE : ae$,

$\therefore$ area $ABCDEF : abcdef$ :: square of AC, or AD, or AE : square of ac, or ad, or ae, respectively.

93. PROP. VIII. *The circumferences of circles are to one another as their radii, or diameters; and their areas are proportional to the squares of those radii, or diameters.*

Suppose *any* two similar regular polygons to have their circumscribing circles drawn about them; these circles will represent *any* two circles. Bisect each of the arcs subtended by each of the sides of the two polygons, and join the points of bisection with the adjacent angular points of the polygons; then two polygons of double the number of sides will be formed, while the circumscribing circles remain the same; and the *perimeters* and *areas* of these latter polygons will obviously approach nearer to the perimeters and areas of the circles than those of the former polygons. Again the arcs subtended by the sides of these polygons may be bisected, and other polygons described with double the number of sides, while the circles remain the same; and so on without limit, until the polygons are made to approach *as near as we please* to the circles.

Now the *perimeters* of similar regular polygons are as the radii of their circumscribing circles, and the *areas* as the squares of those radii, *whatever be the number of sides*, and therefore when that number, as above, is supposed to be indefinitely increased. But, by thus increasing the number of sides the polygons may be made to differ from the circles by less than any assignable magnitude, both as to *perimeter* and *area*. Hence the perimeters, that is, the *circumferences* of the circles will be as their radii, and the *areas* as the squares of those radii.

Also, since the diameters will obviously have the same ratio to each other as the radii, the circumferences of circles will be as their *diameters*, and the areas as the squares of those diameters.

COR. Since circumf. of one circle : circumf. of another :: diameter of the former : diameter of the latter, ∴ *alternately*, circumf. of one : its diameter :: circumf. of the other : *its* diameter; that is, *the ratio of the circumference of every circle to its diameter is the same.*

EXERCISES D.

(1) Define '*hexagon*,' and '*diagonal*' of a polygon. How many different *diagonals* has the *hexagon*?

(2) Define '*angle of* a polygon'; and shew that in *every* polygon the sum of all the angles is a *multiple* of a *right* angle.

(3) Shew that the angle of a regular polygon is always greater than a right angle; and that it increases as the number of sides increases.

(4) Shew that the *angle of* a regular octagon is equal to one right angle and a half. Hence construct a regular octagon upon a given straight line.

(5) Shew that the side of a regular hexagon is equal to the radius of the circumscribing circle.

(6) What is the number of *diagonals* which may be drawn in a polygon of *ten* sides?

(7) Dividing a polygon by means of certain diagonals into the triangles of which it may be supposed to be made up, shew that the *number* of these triangles will always be less by 2 than the number of sides of the polygon.

(8) Shew that in a regular *pentagon* each *diagonal* is parallel to a *side;* and that, if all the diagonals be drawn another regular pentagon will be formed by their intersections within the former one.

(9) Shew that every regular polygon may be divided into equal isosceles triangles. For what polygon are these triangles *equilateral*?

(10) Shew that the *sum* of all the angles of a polygon is not altered by altering the sides either in magnitude or relative position, as long as their *number* remains the same.

(11) Having given a regular polygon of any number of sides, shew how a regular polygon of double the number of sides may be constructed.

(12) State the process by which the *area* of any polygon may be converted into an equivalent rectangle.

(13) Shew that two *similar* polygons are *equal* to one another, if a side of the one be equal to the corresponding side of the other.

(14) If two *similar* polygons be constructed such that a side of the one is ten times the corresponding side of the other, what proportion will the *areas* of the two polygons have to each other?

(15) If you wished to increase a garden, which is in the form of a polygon, so as to become exactly *four* times as large as it is, but to retain its present shape, how would you proceed to lay out the boundary?

(16) Can a circle be made which shall have its circumference *exactly equal* to the circumferences of two other given circles taken together? If so, shew how it may be done.

(17) If the *area* of one circle be *nine* times that of another, what is the ratio of their *diameters*?

(18) Describe a circle whose *circumference* shall be exactly *twice* the circumference of a given circle.

(19) Describe a circle whose *area* shall be exactly twice the area of a given circle.

(20) Shew that the *areas* of circles are to one another as the squares inscribed in them.

(21) Shew that all regular polygons of the same *name* are necessarily *similar*.

(22) The corresponding *sides* of two similar polygons are in the ratio of a side of a square to its diagonal; find the ratio of the *areas* of the polygons.

(23) If in any circle four radii be drawn at right angles to one another, and with each of these four radii as *diameters* circles be described within the former, shew that the areas of the four circles are together equal to that of the original circle.

(24) From a given polygon cut off a *similar* polygon whose area shall be *one-fourth* of the original one.

(25) Shew how the *square* may be found which is equal to any given polygon.

END OF PART I.

ADVERTISEMENT TO PART II.

In the following Part I have further prosecuted my design of separating the *Art* from the *Science* of Geometry. It should not be forgotten, however, that this separation is merely a matter of *arrangement*, with a view to making the learner's course more precise than heretofore, and affording him a better footing as he proceeds. Hitherto the practice has been, for the most part, in this country, to teach the *Science* to one class, and the *Art* to another— so that, whilst the Students of our Universities have cared little for the Art, the pupils of our Commercial Schools have cared less for the Science. It seemed to me, that this divorcement of practice and theory was both unsatisfactory and unnecessary; and that no good reason can be alleged, why either the University Student's excellent knowledge should fail, as it has done, to fix a distinct impress upon practical Art, or the artisan's skilled workmanship be constantly marred by the violence done to the true principles of Science. My intention has been, therefore, to do something towards bringing Art and Science together again, so far as to make them better friends, not by jumbling the two together, but by assigning to each its distinct duty, and so placing them that they *must* mutually assist each other. Accordingly, although admitting the value of good instruments and a dexterous handling of them, I have never in a single instance in the following Part supposed the fingers to

work without the head. How far I may be able, in the prosecution of my design, to effect a breach in the present style of popular education, fortified as it is by custom and prejudice, I know not; but perhaps it may provoke some educators at least to a wholesome jealousy to be told, that for every book published in England during the last 20 years, *combining Art and Science for the use of the middle class and artisans,* not less, I believe, than 20 such books have been published both in France and Germany.

T. L.

MORTON RECTORY, ALFRETON,
Jan. 31, 1855.

CONTENTS.

ELEMENTS OF

GEOMETRY AND MENSURATION.

PART II.

GEOMETRY AS AN ART.

94. Geometry as an art is the practical application of 'Geometry as a Science', and is sometimes called '*Practical* Geometry', by which is meant Geometry *in Practice.*

This practical application consists in *doing* those things, which in Part I. it has been shewn *may* be done, according to strictly defined geometrical notions and principles. For example, in the former Part a *square* was defined to be "a parallelogram, which has all its sides equal and all its angles right angles"; in this Part such a construction is required to be *actually made* under certain given circumstances. Also, generally, the *Propositions* demonstrated in the former Part are in this required to be known, and *put to use, for purposes of Construction* and *Design;* and that without any respect to order or precedence, such Proposition being always taken, *wherever* it may stand in the former Part, as we judge will most readily and efficiently serve our purpose in this.

95. In strict Geometry, be it remembered, a *point* has no magnitude, neither *length,* nor *breadth,* nor *thickness.* A *line* also has *length* only, and neither *breadth,* nor *thickness.* And, *in practice,* the nearer we can bring our *points* and *lines* to these definitions the more strictly correct will be the work depending upon them. For, if that, which should be a fine *point,* be in fact a *circle* of considerable size, then in measuring from such a point, or in joining two such points by a line, it is obvious that we should be liable to considerable error. In like man-

ner, if we make *lines* broad and coarse instead of fine, then, in the case of such lines intersecting each other, the *points* of intersection cannot be *accurately* marked, and therefore plainly any measurements from such points will be subject to error. And so in other cases, where *points* and *lines* require to be actually *traced.*

Hence, although *perfect* accuracy is really unattainable, it is plain, that, in the application of Geometry to practical art or design, correctness of construction is *most nearly* attained where the precision of the Geometrical Definitions is most closely regarded.

96. But before Geometry can be put in *practice*, certain TOOLS or INSTRUMENTS are required, of which we will here give a short description:—

(1) The POINTED PENCIL, or PEN, or other *marker*, is used to trace out *lines* and to mark *points* on paper, or board, or other surface. It is only requisite for accurate workmanship that the marking point be kept as *fine* as possible.

(2) The FLAT-RULER, or STRAIGHT-EDGE, is used for drawing *straight lines* on a given *plane surface;* and for determining whether lines already drawn be straight; and for some other purposes. It is made of various substances, but generally of wood, the only essential requisite being that it shall have one *edge*, or boundary, *throughout its whole length, perfectly straight.* This being the case, it is clear that a straight line may be drawn on any given plane surface by placing the *straight-edge* in contact with the surface, and drawing the pencil or other marker carefully along it. And a given straight line may be *tested* as to its straightness, by placing the *straight-edge* close along-side the line, and observing whether the two coincide or not with each other.

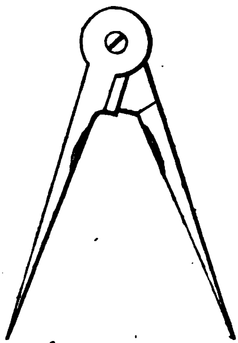
Of course, if the *ruler* itself be not perfectly straight, it cannot be used to any good purpose, where accuracy of construction is required. But this fault, if it exist, is easily detected by the following simple method:—

Place the *straight-edge* in close contact with any *plane surface,* as paper or board, and draw a straight line along it in the usual way, *to the whole extent of the straight-edge.* Then turn the straight-edge round so that its extremities exactly change places, and draw a

straight line along it again. If the two lines thus drawn *coincide throughout their whole extent* the *ruler* is correct; but otherwise not.

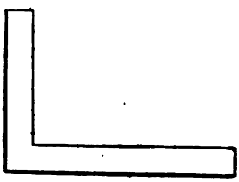
(3) The COMPASSES consist of two equal *legs* connected together by a hinge or joint at one end of each, and having the other ends worked down to fine points, which meet closely when the *legs* are brought into contact, that is, when the compasses are shut. The hinge-joint works rather *stiffly*, so that the legs, when left to themselves, may remain fixed at any angle by which we may choose to separate them.

This instrument is used for measuring off short distances, that is, *straight lines*; and also, when a portion of one leg is moveable, and replaced by a pen or pencil, for drawing small circles. It is obvious, that with such an instrument a circle of any *given radius*, within certain limits, may be traced. For, if the legs be separated so that the distance between their extreme points is equal to the given radius, then by *fixing* one point in the paper or board and causing the compasses to revolve round it, the other point, being kept in contact with the paper or board, will evidently trace out the required circle.



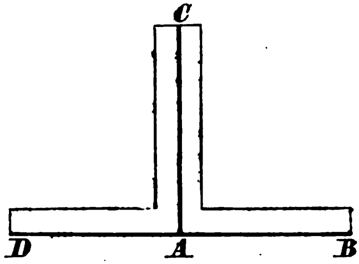
(4) The SQUARE consists of two *flat-rulers* firmly connected together in such a manner that both their inner and outer edges are at *right angles* to each other.

This instrument is used chiefly by masons and carpenters for constructing *right angles*, and for testing the correctness of angles which ought to be *right* angles.



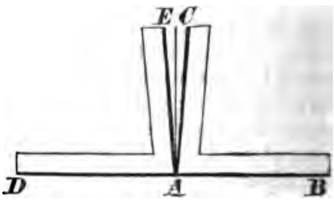
Whether the *square* itself be correct or not, may easily be determined by the following method:

1st. To try the *outer* edge; on any plane surface trace, by means of it, the angle BAC; extend BA in the same straight line to D. Then turn the *square* round the point A, so that the outer edge which before coincided with AC now coincides with AD. If then the outer edge of the other limb exactly coincides with AC, the *square* is correct as to its outer edge; otherwise not.

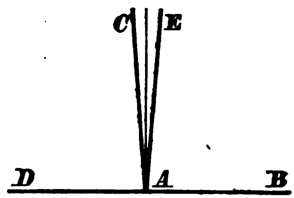


2nd. To try the *inner* edge; proceed in the same manner, only making use of the *inner* edge where before the *outer* was used.

By this method, also, the *amount* and *quality* of the error, if any, is ascertained. For, if the error be *in defect*, that is, if its angle be *less than* a *right* angle, the *square* will appear, in the two positions above mentioned, as in the annexed fig., where the angles BAC, DAE are equal, and together fall short of *two* right angles by the angle CAE. Therefore the *error* of the square is equal to half the angle CAE.

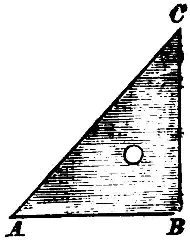


Similarly, if the error be *in excess*, that is, if the angle of the square be *greater than* a right angle, the angles drawn in the two positions of the square will *over-lap* each other, as in the angles BAC, DAE in the annexed fig.; so that the two angles together *exceed* two right angles by the angle CAE; and since they are equal to each other,



therefore, in this case also, the error of the square is equal to half the angle CAE.

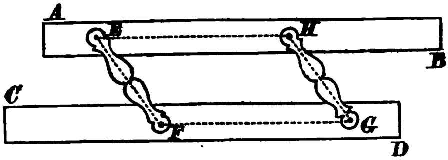
(5) The draughtsman's *Triangle* is simply a thin triangular piece of wood or ivory, with its sides accurately and smoothly made, so that any one of them may be used as a *ruler*, and two of them, as AB, BC, forming a *right angle*.



A small hole is cut through the instrument, that it may be handled and moved along the surface of the paper or board more easily.

This instrument is used, as it obviously may be, for drawing lines at *right angles*, or *perpendicular*, to other lines; and also for some other purposes, as will appear hereafter.

(6) The PARALLEL-RULER consists of two *flat-rulers*, similar and equal in all respects, as AB, CD which are so connected together by means of two equal pieces of brass, EF, GH, working loosely round fixed pins in the rulers at the points E, F, G, H, that, when the rulers are separated, both their outer and inner edges are *parallel* to each other.



It is requisite, not only that EF should be equal to GH, but also that the distance EH between the pins in one ruler be equal to FG the distance between the pins in the other. In which case the lines joining the points E, F, G, H always form a *parallelogram* (40); and as these points are equidistant from both the outer and inner edges in each ruler, those *edges* will always be *parallel*.

Hence it is plain, this instrument may be used for drawing any number of *parallel straight lines*, or for drawing one or more straight lines parallel to a *straight line already drawn*.

(7) A SCALE OF EQUAL PARTS is mostly a *Flat-*

ruler which has its whole *length* divided into a certain number of equal parts, and each of these parts again subdivided into smaller equal parts, the several points of division being marked by lines across the *face* of the ruler.

The common *foot-rule* is an example of a *Scale of Equal Parts*, its length being divided into 12 equal parts called inches, and each inch into *parts* of an inch.

This instrument is used for comparing one length or straight line with another.

97. The above are the instruments which are in most common use for Geometrical purposes. Others, more complex, will be described hereafter.

It seems only necessary to observe here, that the workman or draughtsman is much to be blamed, who is content to work with *faulty* tools or instruments, when he is able to procure better; seeing that very small errors, will often, *by multiplication*, produce seriously defective results. In such instruments as the foot-rule, and square, this will be obvious to the most common understanding.

N.B. The Definitions and Propositions of Part I. are all *assumed* in this Part.

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98. Proposition I. *To draw a straight line on a plane surface between any two given points.*

(1) This is mostly done, if the given points be not too widely apart, by means of a *flat-ruler*, or *straight-edge*. The ruler is placed so as to have the same edge exactly on both the points, and a fine pen, or pencil, is drawn carefully along it in contact with both the straight edge and the surface on which the line is required to be drawn. See (96).

(2) But, if the given points be so far apart that the *ruler* is insufficient, then other modes are adopted according to circumstances. Thus, it is known that *light* always travels, if uninterrupted, in a straight line from one point to another, and consequently any workman is readily able to determine a point *C* intermediate to *A* and *B*,

the two given points, which shall be in the same straight line with A and B. He places his eye at A, so as to see B, and marks a point C which appears to eclipse, or coincide with, B. Then he can join A and C, and also B and C, by means of his ruler, or straight-edge; and the thing required is done.

(3) In some cases where the given points A and B are *very* distant, it may be necessary to lay down, by the eye, *several* intermediate points, C, D, E, &c., and by joining each contiguous pair, one continuous straight line will be traced from A to B.

(4) Another mode, adopted mostly by sawyers, for marking out the course of the saw, is to stretch *tightly* between the points a thin cord which has been chalked throughout its whole length or dipped in some marking material, and then, while the ends are kept fixed, the cord is drawn a little from the surface of the wood and allowed to recoil with force back again, whereby a distinct line is traced between the two ends and sufficiently straight for practical purposes.

Gardeners, bricklayers, and others also, make use of a tightly stretched cord for determining the straight line which lies between any two given points.

But in all cases where a *cord* is used it must *lie along* the plane surface, on which the straight line is to be drawn, throughout its whole extent, otherwise its own *weight* will cause it to deviate from the *straight* line joining its extremities.

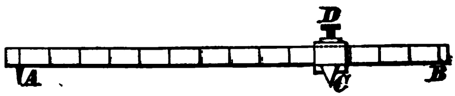
99. Prop. II. *To draw a circle on a plane surface, about a given point in it as its centre, and with a radius equal to a given straight line.*

(1) This may easily be done, within certain limits, by means of the ordinary *compasses*. Open the legs until their extreme points exactly coincide with the extremities of the given line; then fix the foot of one leg on the point which is to be the centre, and by making the compasses to revolve round this point while the foot of the other leg is kept in contact with the surface on which the circle is to be drawn, the latter will trace out the circle required.

The main thing to be attended to in this operation is,

that the angle by which the legs of the compasses are separated does not *vary* throughout it; for any change in this angle will obviously produce a corresponding change in the *radius* of the circle. Such an error, if it existed, would generally be discovered from the fact of the circumference not returning into itself; but especial care is needed in this respect, whenever, as is often the case, only *arcs* are drawn, instead of the whole *circumferences*.

(2) When the *radius* of the circle is greater than the distance by which the points of the *ordinary compasses* can conveniently be separated, another instrument is used called BEAM-COMPASSES. This instrument consists of a beam or bar, *AB*, in the lower side of which, near one extremity is *fixed* a steel point, as *A*; and another point, *C*, is fixed to a clamp which *slides* along *AB*, and may be held tight by means of the screw *D* at any proposed distance from *A*. The beam beginning from the fixed point *A* is usually graduated into inches and parts of an inch.



In using the instrument, *AC* is first made equal to the given radius, and the screw *D* made tight; then the steel point *A* is placed upon the point which is given for the centre of the circle; and while *A* is kept upon that point, the beam is made to travel round it, having *C* in contact with the surface on which the circle is to be traced. It is plain, that the circle thus traced by *C* is the circle required.

The advantages of having the beam *graduated* is obvious, because we are thus enabled, *without the aid of any other instrument*, to describe a circle of any proposed *radius*, not exceeding the length of the beam, expressed in inches and parts of an inch.

(3) When the radius of the circle is still greater and beyond the power of the *Beam-Compasses*, the circle may be traced by means of a *cord*, having a small loop at each end, and equal in length to the given radius. A pin, or nail, or peg, according to circumstances, is passed through one loop and fixed firmly on the surface on

which the circle is to be traced. A marker is then passed through the other loop, and made to travel round in contact with the surface, while the cord is kept perfectly tight.

100. Prop. III. *From a given straight line to cut off a part equal to another given straight line.*

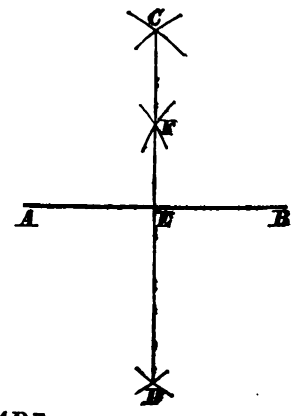
(1) If the smaller of the two lines be within the range of ordinary compasses, this is readily done. It is only necessary to open the compasses, until they exactly embrace the lesser line, and then to transfer it to the greater line by placing one foot of the compasses, at one *extremity* of the greater line and marking the point where the other foot meets it.

(2) If the smaller line exceed the range of the compasses, its length may be marked by placing along it a straight rod, or rule, or tight cord, and then applying this *measure* of the smaller line to the greater, a part may be readily cut off from the latter equal to that *measure*, that is, equal to the smaller line.

101. Prop. IV. *To bisect a given straight line, that is, to divide it into two equal parts.*

This is done with theoretical exactness in (27); but *in practice* such a method would never be adopted. The same thing may be readily done thus:

(1) Let AB be the given line; with centres A and B, and radius as great as is convenient, draw two pairs of intersecting arcs at C and D, on opposite sides of AB. Then join CD cutting AB in E; and AB is *bisected* in E. Or, if it be inconvenient to form intersecting arcs on *two* sides of AB, diminish the opening of the compasses, and draw both pairs on the *same* side, as at C and F. Join CF, and *produce* it to meet AB in E.



[Not only does CE *bisect* AB in E, but it is also at the same time *at right angles* to AB.]

(2) The following method will serve for most purposes:

AB the given line; place one foot of the compasses on the point A, and the other on a point C in AB as nearly half-way between A and B as you can guess; turn the compasses round C, and if the foot which was at A is found to fall exactly on B, the thing is done, because in this case $AC = CB$. But if not, mark the point D in AB or AB produced, where the first foot meets it, so that $AC = CD$; and while the other foot is held firmly at C extend the former to E, the middle point, as near as you can guess, of BD. With this opening of the compasses mark off $BF = CE$. And if E has been correctly taken the middle point of BD, F will be the middle point of AB, as will easily be seen.

If, however, the middle point of BD has not been correctly marked (as it generally may be without any sensible error in *very short* lines), the process will have to be repeated with a still smaller line, representing the difference between AF and BF; and so on, the line to be bisected *at sight* continually growing less. But with tolerable care a person who *wishes* to bisect a given line, will, by this method, speedily do it.

(3) Another method very commonly used, but partly *arithmetical*, is as follows:—

To the given line apply *a Scale of equal Parts*, and note the number of its divisions over which the whole line extends. (This will, of course, be most easily done by making the *beginning* of the scale to coincide with one extremity of the line). Divide this *number* by 2, and mark the point where the resulting quotient is found on the scale in contact with the line; that point will bisect the given line as required.

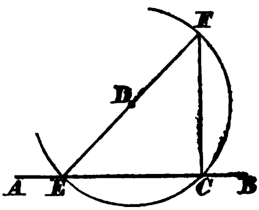
According to this method the Carpenter, or Builder, divides any straight line or length into two equal parts by means of his *Foot-Rule*, or *Tape*. The method is especially applicable to *long* lines extending beyond the span of the ordinary *compasses*.

102. PROP. V. *To draw a straight line* at right angles *to a given straight line from a given point in it.*

This is done with theoretical exactness in (28); but the method there employed is not practically applicable

in every case; for example, it cannot be used when the given point is at, or very near to, the *end* of the given line, and the line from its position cannot be '*produced*'.

(1) In such a case, (and the same method is *generally* applicable) let AB be the given straight line, and C the given point in it, from which it is required to draw a straight line *at right angles* to AB. Take a point D about equidistant, at sight, from AC and the required line; with centre D and radius DC describe a circle meeting AC in E; join ED, and produce ED to meet the circle in F; then join FC; FC is at right angles to AB.



For, since EF is a diameter, ECF is a *semi-circle*, and the '*angle in*' a semi-circle is a *right angle* (54), $\therefore \angle ECF$ is a *right* angle.

The *Draughtsman* will generally employ a more expeditious method than the preceding.

(2) Either he will use his 'Triangle' as a ruler (96), placing it so that its side AB coincides with the given line, and the point B is on the given point from which the line at right angles to the given line is required to be drawn; and then draw from the point B a straight line along BC, which will be the line required.

(3) Or, provided with a thin *Flat-ruler*, which has a line accurately marked across one of its faces *at right angles* to both its parallel edges, he will place the ruler so that that line lies exactly over the given line, and with one extremity on the *given point*. He will then draw a line along that edge of the ruler, and it will be the straight line required.

In practice this is, perhaps, with a correct ruler, the most accurate of all methods; and, if the line, *as given*, be *shorter* than the cross line traced on the ruler, it may easily be '*produced*' to begin with, until it is of sufficient length to shew itself on opposite edges of the ruler.

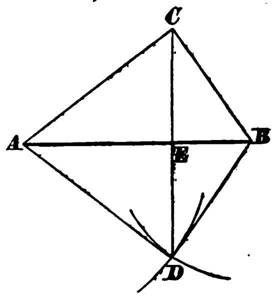
(4) The *Carpenter* and *Mason* will generally apply the '*Square*' for this purpose in a way which needs no explanation, when the '*Square*' itself is understood. See (96).

Another method, requiring a knowledge of *Mensuration*, will be given in Part III.

103. Prop. VI. *To draw a straight line* perpendicular *to a given straight line through a given point* without it.

Let AB be the given straight line; C the given point without it, from which it is required to draw a straight line *perpendicular to AB.*

(1) With centre A, and radius AC, describe a short arc on the other side of AB, and as nearly opposite as you can guess, to C. With centre B, and radius BC, describe another arc intersecting the former in the point D. Then join CD meeting AB in E, and CE is the *perpendicular* required.

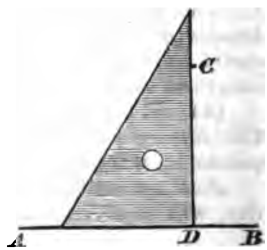


For, joining AC, AD, BC, BD, the triangles ABC, ABD are equal in all respects; $\therefore \angle CAE = \angle DAE$. Also $\angle ACE = \angle ADE$, since $AC = AD$ (26); $\therefore$ the remaining $\angle AEC$ = the remaining $\angle AED$ (37), and $\therefore \angle AEC$ is a *right* angle, that is, CE is perpendicular to AB.

(2) *Another Method.* See the fig. in (102); let AB be the given line, and F the given point without it. Draw FE to any point E, taken at random, in AB. Bisect FE in D; with centre D and radius DE describe the semicircle ECF cutting AB in C. Join FC, and it is the *perpendicular* required.

(3) The *draughtsman* will generally use his '*Triangle*', or '*Flat-ruler*' with cross line, as in (102), for this purpose.

He will place the *Triangle* so that one of the sides forming the right angle lies along AB; slide it towards the given point C, keeping the former side carefully on AB; and when the other side passes through C, draw the line CD along it to meet AB in D. CD is the perpendicular required.



Or, if C be too far distant

from *AB* for the *Triangle*, he will make use of the *Flat-ruler* with cross line, precisely as in (102).

(4) The Carpenter and Mason will apply the '*Square*' in a similar manner.

104. PROP. VII. *Through a given point to draw a straight line which shall be* parallel *to a given straight line.*

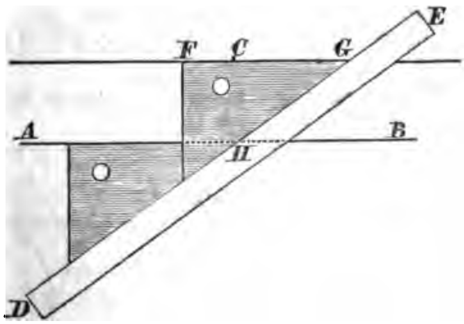
(1) This is done in *two* ways in (36); and the latter method is sufficiently practical with no other instrument than the ordinary '*Square*', or '*Triangle*'.

(2) But the same thing is most readily done by means of the '*Parallel-Ruler*', made for the purpose, unless the given point be at a greater distance from the given line than the extreme width of the *ruler* when opened to its fullest extent.

If the given point be *nearer* to the line than the width of the ruler when *closed*, place *the whole ruler* on *the other side* of the given line with one edge exactly along the line; then move this edge, while the other half of the ruler is *held tight*, until it exactly passes through the given point, and draw the required line along this edge in that position.

If the point be at a distance from the line somewhat greater than the width of the ruler when closed, place the closed ruler *between* the point and the line, with one outer edge coinciding with the line; *hold this tight*, and move the other outer edge until it passes through the point; then draw along it the line required.

(3) *Another method* is by means of the '*Triangle*' and '*Flat-ruler*'.

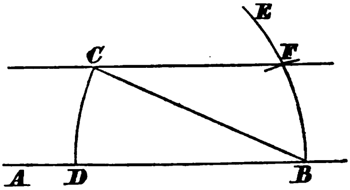


Let AB be the given straight line, and C the given point.

Place the *Triangle* on the opposite side of AB to C, so as to have one of the sides forming the right angle exactly on AB. Then lay the Ruler along the hypothenuse of the *Triangle*, as DE in the annexed fig., and while the Ruler is *held tight* in that position, slide the Triangle along it, until the same side which was on AB passes through C. Then draw FCG along that side, and it shall be parallel to AB, and may be *produced* both ways from F and G as required.

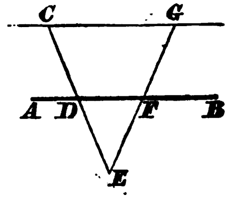
FCG is *parallel* to AB, because they are two straight lines intersected by another straight line DE, making the exterior angle AHD equal to the interior and opposite angle FGH on the same side of it (34 Cor. 1).

(4) *Another Method.* With centre B, *any point* in the given line distant from A, and radius BC, describe the arc CD cutting AB in D. With centre D, and the same radius as before, describe the arc BE on the same side of AB; and from BE measure off, with the compasses or otherwise, BF equal to CD, remembering that in circles of the same radius equal *chords* subtend equal *arcs* (58). Join CF, and it is *parallel* to AB.



For CD, BF being equal arcs of equal circles, they subtend equal angles at the centre (59), that is, the *alternate* angles CBD, BCF are equal, ∴ CF is parallel to AB (34).

(5) *Another Method.* From the point C draw any straight line CE meeting AB in D; and make $DE = CD$. From E draw another line EG cutting AB in F, and make $FG = EF$. Join CG, and CG is the line required.

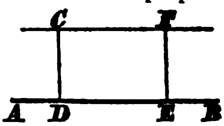


For the two sides EC, EG of the triangle ECG are divided *pro-*

portionally in the points D and F, $\therefore$ CG is *parallel* to DF (70).

(6) The Carpenter or Mason will employ a method still more simple:—

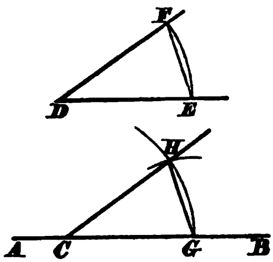
By means of his '*Square*' he will draw CD *perpendicular* to AB; and from *any* other point E in AB he will draw EF *at right angles* to AB, making $EF = CD$. Then join CF, and CF is *parallel* to AB (35 Cor.)



It is upon this principle of *equidistance* of parallel lines that the instrument called a *Joiners' Gauge* is constructed.

105. Prop. VIII. *From a given point in a given straight line to draw another straight line making with the former an angle equal to a given angle.*

(1) Let AB be the given straight line; C the given point in it; and EDF the given angle. (The angle is here supposed to be *given* by being traced on the same plane as that on which the other angle is to be drawn.) With centre D and any convenient radius, the greater the better, describe the arc EF. With centre C, and the same radius, describe the arc GH. Then with centre G and radius equal to the *chord* EF, describe an arc cutting GH in H. Join CH, and it shall be the straight line required.

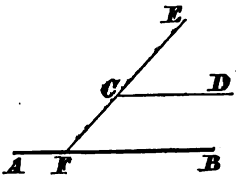


For, if EF, GH be joined, they are equal chords of equal circles, $\therefore$ the *arcs* EF, GH are equal, and $\therefore$ $\angle GCH = \angle EDF$ (59).

(2) If either of the lines which form the given angle, EDF, be in the same straight line with, or parallel to, the given line, AB, then it will only be necessary to draw through C, (by means of the *Parallel-Ruler* or otherwise), a straight line *parallel* to the *other* side of the angle EDF.

106. Prop. IX. *Through a given point* without *a given straight line to draw another straight line making with the former an angle equal to a given angle.*

Let AB be the given straight line, and C the given point. Through C draw CD parallel to AB. From C draw CE making with CD the angle DCE equal to the given angle (105). Produce EC to meet AB in F, and EF is the line required.



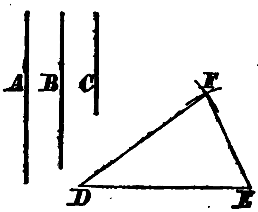
For the straight line EF meets the *parallel* lines AB, CD;
$\therefore \angle BFC = \angle DCE =$ the given angle (34).

107. Prop. X. *To construct an* equilateral *triangle having each of its sides equal to a given straight line.*

This is done in (23); and no better method can be adopted in *practice*.

108. Prop. XI. *To construct a triangle with its sides respectively equal to three given straight lines.*

Let A, B, C, be the three given straight lines. Take DE equal to A. With centre D and radius equal to B describe an arc of a circle, on that side of DE on which the triangle is to be drawn; and with centre E and radius equal to C describe another arc on the same side of DE intersecting the former arc in F. Join DF, EF; and DEF is plainly the triangle required.



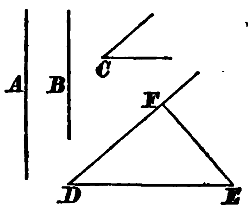
N.B. Since in *every* triangle any two sides are together greater than the third side (38), the given straight lines must be such that any two of them are greater than the third.

Cor. If $A = B = C$, the same construction will hold, and the triangle is *equilateral*. If $B = C$, the triangle is *isosceles*.

109. Prop. XII. *To construct a triangle with two sides respectively equal to two given straight lines and an angle equal to a given angle.*

Let A, B, be the given straight lines, and C the given angle.

(1) In the case when the angle C is to be between the given sides, draw $DE=A$; at the point D make $\angle EDF=\angle C$ (105), and $DF=B$. Join EF, and DEF is the triangle required.

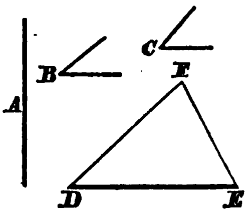


(2) In the case when the angle C is to be *opposite* to one of the given sides, as B, proceed as before, but instead of cutting off DF equal to B, with centre E and radius B describe an arc cutting DF in F. Then join EF, and DEF is the triangle required.

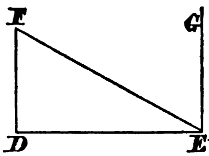
110. Prop. XIII. *To construct a triangle with two angles respectively equal to two given angles, and one side equal to a given straight line.*

Let A be the given straight line, and B, C, the two given angles.

(1) In the case when the given side is to be *adjacent* to *both* the given angles, take $DE=A$; make $\angle EDF=B$ (105), and $\angle DEF=C$. Then DEF is the triangle required.



(2) In the case when the given side is to be *opposite* to one of the given angles, take $DE=A$, make $\angle DEF=B$: through E draw EG making $\angle GEF=C$; then through D draw DF parallel to EG, and DEF is the triangle required. For $\angle DFE=\angle FEG=\angle C$ (34).



Obs. In every triangle there are six parts, *three* sides and *three* angles; and of these six if *any three* be given, except three *angles*, the triangle is determined. But, it is plain that a triangle is not known, when its *angles only* are known, because an infinite number of different triangles may have the same or equal angles, as will easily appear by drawing *within* any given triangle straight lines parallel to the sides.

111. Prop. XIV. *To construct a* right-angled *triangle with given parts.*

1st. When each of the two *sides* forming the right angle are given. The construction in this case is too obvious to require even to be stated.

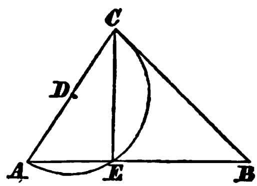
2nd. When one *side,* and the *adjacent* acute *angle,* are given. From one end of the given side, draw a straight line at right angles to it, and from the other end, draw another straight line making with the given side an angle equal to the given angle (105). These two lines, with the given line, will form the required triangle.

3rd. When one *side,* and the *opposite angle* is given. See fig. 2 in (110); let DE be the given side; from D and E draw DF, EG at right angles to DE; make $\angle GEF$ = the given angle; DEF is the required triangle. For DF is *parallel* to EG, $\therefore \angle DFE = \angle FEG =$ the given angle.

4th. When one *side,* and the hypothenuse, are given. Upon the hypothenuse as a diameter describe a semicircle. From one end draw a *chord* equal to the given side; and join the other end with the end of that chord.

112. Prop. XV. *From the vertex of one of the angles of a given triangle to draw a* perpendicular *upon the opposite side.*

Let ABC be the given triangle. It is required to draw from the point C a perpendicular to AB.



Bisect AC in D; with centre D and radius DA or DC describe the semi-circle AEC, intersecting AB in E. Join CE, and it is the perpendicular required.

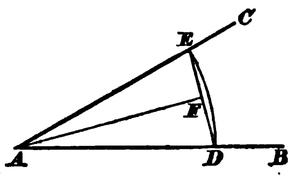
For $\angle CEA$ is an '*angle in a semi-circle*', and is therefore a right angle (54).

The same thing may, of course, be done by *any* of the methods given in (103) for drawing a straight line perpendicular to a given straight line from a given point without it.

113. Prop. XVI. *To bisect a given angle, that is, to divide it into two equal angles.*

(1) Let BAC be the given angle. With centre A, and

any convenient radius (the greater the better) describe an arc cutting AB in D, and AC in E; join DE, bisect DE in F, and join AF. Then AF bisects the $\angle BAC$.

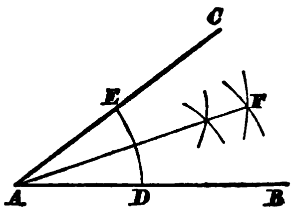


For it may easily be shewn that $\angle EAF = \angle DAF$ (24).

(2) Most persons, in practice, knowing that in the same circle equal arcs subtend equal angles at the centre, instead of bisecting the *chord* DE, would bisect the *arc* DE, and join the point of bisection and the point A. And the *arc* DE would mostly be bisected *by trial* with the compasses, since equal *chords* subtend equal *arcs*.

(3) The following is an expeditious method of bisecting an angle :—

With centre A and any convenient radius describe the arc DE; then with the *same radius* and centres D and E describe arcs intersecting in F; and join AF. AF bisects the angle BAC.



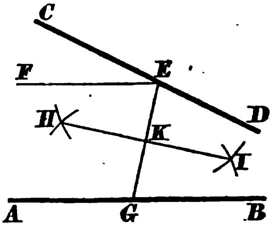
The accuracy of the work may also readily be tested. For, diminishing the opening of the compasses, if with centres D and E another pair of intersecting arcs be drawn, their point of intersection ought to be in AF.

(4) *Another Method* by means of the *Parallel-Ruler*. Take any point D in AB; through D draw a straight line DF parallel to AC, and make $DF = AD$, and join AF. The angle BAC is bisected by AF (26 and 34).

114. Prop. XVII. *To bisect the angle between two given straight lines, when the vertex of the angle is not given, and cannot conveniently be determined.*

Let AB, CD be the two given straight lines, which cannot conveniently be produced to meet. From any point E in one of them, CD, draw EF parallel to the other, AB. Bisect $\angle DEF$ by the straight line EG (113), meeting AB in G. With centres E and G, and any radius,

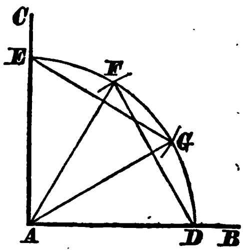
draw two pairs of intersecting arcs, and draw HI joining the points of intersection; then HI is the line required.



For, by construction, $\angle DEG = \angle FEG$. Also $\angle FEG = \angle EGB$, since FE is parallel to AB (34); $\therefore \angle DEG = \angle EGB$, and $\therefore$ the triangle which would be formed by producing the given lines to meet would be *isosceles* upon base EG. Also HI *bisects* that base *at right angles* (101), and every straight line which *bisects* the base of an *isosceles* triangle *at right angles* will, if produced, pass through the *vertex* of the opposite angle, as may easily be proved.

115. Prop. XVIII. *To trisect a right angle, that is, to divide it into three equal angles.*

Let BAC be the given right angle; with centre A, and any radius AD, a part of AB, (the larger the better), describe an arc of a circle meeting AB in D, and AC in E; with centre D, and the same radius as before, draw a small arc to intersect the former in F; and, again, with centre E, and radius as before, another small arc to intersect the first in G. Join AF, AG; and the thing required is done.

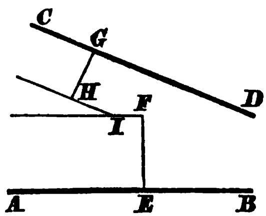


For, joining DF, and EG, AFD, and AEG are both *equilateral* triangles; $\therefore \angle DAF =$ *one-third* of two right angles (26 Cor. and 37,) that is, *two-thirds* of *one* right angle, and $\therefore \angle EAF$, which makes up the right angle, must be *one-third* of a right angle. Similarly, since AEG is an equilateral triangle, $\angle EAG =$ *two-thirds* of a right angle, and $\therefore \angle DAG =$ *one-third* of a right angle. Hence $\angle FAG$ which, added to these two, makes up the right angle, must also be *one-third* of a right angle.

Cor. Hence also a *quadrant*, or the *arc* of a quadrant, may be divided into three equal parts.

116. Prop. XIX. *To find the point which is at certain given distances from two given straight lines in the same plane.*

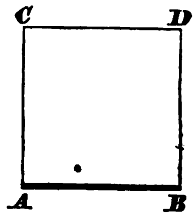
Let AB, and CD be the two given straight lines. Take any point E in AB; draw EF at right angles to AB, and equal to the given distance from AB. Also from any point G in CD draw GH at right angles to CD, and equal to the other given distance. Through F draw FI parallel to AB, and through H draw HI parallel to CD; and the point of intersection, I, of these parallels is the point required.



For, drawing through I two straight lines parallel to EF, GH, the proof is obvious by (40).

117. Prop. XX. *To construct a square with each of its sides equal to a given straight line.*

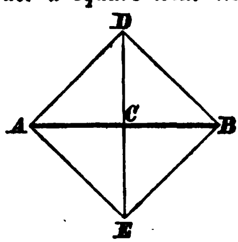
This is done in (42); but in practice it will most frequently be done thus:—Take AB equal to the given length or line; from A and B, by means of the '*Square*', or the '*Triangle*', draw AC, BD, *at right angles* to AB; and make each of them *equal to* AB. Then join CD; and $ACDB$ is the square required.



The correctness of the work may always be easily tested by applying the *compasses* or *rule* to the opposite corners, for in a true square the *diagonals* must be *equal* to one another, as well as the *sides*.

118. Prop. XXI. *To construct a square with its diagonal equal to a given straight line or length.*

Let AB be taken equal to the given straight line; with centres A and B, and any convenient radius greater than the half of AB, draw two pairs of intersecting arcs on opposite sides of AB; join the points of intersection by

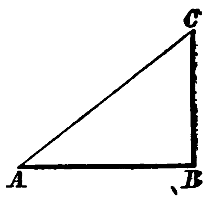


a straight line cutting AB in C. In this line take CD, CE, each equal to CA or CB; join AD, BD, AE, BE, and $ADBE$ is the *square* required.

For, if with centre C and radius CA a circle be described, it will pass through the points A, D, B, E; and each of the angles of the fig. $ADEB$ will be the 'angle in a semi-circle', and ∴ a *right angle* (54). Also the sides will be the chords of equal arcs, and ∴ will be equal (58).

119. Prop. XXII. *To construct a square which shall be equal to the sum of two given squares.*

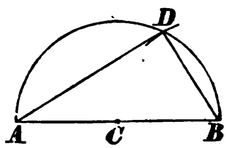
Draw the straight line AB equal to a side of one of the given squares, and BC, at right angles to AB, equal to a side of the other; and join AC. Then construct a square whose side is equal to AC (117), and it will be the square required.



For, since $\angle ABC$ is a *right* angle, the square of AC = square of AB + square of BC (43).

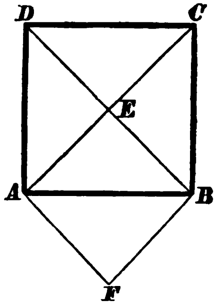
120. Prop. XXIII. *To construct a square which shall be equal to the difference of two given squares.*

Draw the straight line AB equal to a side of the greater of the given squares; bisect it in C; with centre C and radius CA describe a semi-circle. With centre B, and radius equal to a side of the other square, describe a small arc intersecting the semi-circle in D. Join AD. Then construct a square whose side is equal to AD (117), and it will be the square required.



For, joining BD, ADB is a right-angled triangle (54), ∴ square of AB = square of AD + square of BD; and taking from these equals the square of BD, we have the difference of the squares of AB and BD = the square of AD.

121. Prop. XXIV. *To construct a square which shall be half of a given square.*

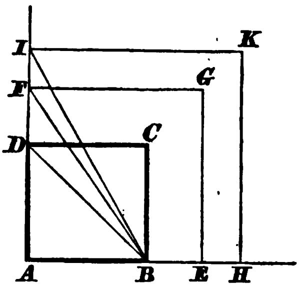


Let $ABCD$ be the given square. Draw the diagonals AC, BD, intersecting in E. Through A draw AF parallel to BE; and through B draw BF parallel to AE. Then $AEBF$ is the square required.

For, by construction it is a *parallelogram;* and one of its angles, viz. $\angle AEB$, is a *right* angle, since the triangles AEB, AED are equal in all respects; also $AE=EB$; $\therefore$ *all* its angles are *right* angles, (42 Cor. 2) and *all* its sides *equal* (40). It is also equal to the two triangles AEB, AED, which are together half the square $ABCD$.

122. Prop. XXV. *To construct a square which shall be any* multiple *of a given square.*

Let $ABCD$ be the given square; produce AB, AD, indefinitely towards B and D, join BD; in AB produced take $AE=BD$; and in AD produced $AF=BD$. Draw FG parallel to AB, and EG parallel to AD. Then $AEGF$ is a square, and it is *double* of the square $ABCD$.



Again, join BF; take $AH=BF$, and $AI=BF$; complete the square $AHKI$, and it is *three times* the square $ABCD$; and so on for succeeding multiples.

For, $AEGF$ = square of AE,
= square of BD,
= square of AB + square of AD (43),
= *twice* the square of AB,
= *twice* $ABCD$.

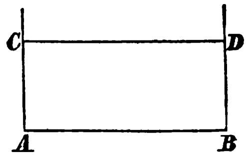
Also, $AHKI$ = square of AH,
= square of BF,
= square of AB + square of AF,
= square of AB + *twice* square of AB,
= *three times* the square of AB;

and so on for succeeding multiples.

123. Prop. XXVI. *To construct a rectangle with sides equal to given straight lines.*

Since the opposite sides of every rectangle are equal to one another (13), *two* straight lines only are needed to be given in this case.

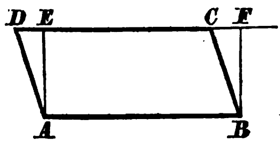
Draw the straight line AB equal to one of the given lines, and from the extreme points in it draw AC and BD at right angles to AB, making each of them equal to the other given line; and join CD. Then $ACDB$ is obviously the rectangle required.



The *correctness* of the work may be *tested*, as in the square, by measuring the *diagonals* AD, BC, which ought to be equal to one another.

124. Prop. XXVII. *To construct a rectangle which shall be equal to a given parallelogram.*

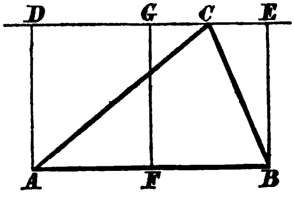
Let $ABCD$ be the given parallelogram; produce the side DC; and from the extreme points of the base AB, draw AE, BF perpendicular to DC, and DC produced. Then $ABFE$ is the rectangle required.



For $ABFE$ is a *parallelogram*, and $ABFE$, $ABCD$ are upon the same base AB, and between the same parallels (41).

125. Prop. XXVIII. *To construct a rectangle which shall be equal to a given triangle.*

Let ABC be the given triangle; through C draw DCE parallel to AB; and through A and B draw AD, BE at right angles to AB; bisect AB in F, and through F draw FG at right angles to AB; $AFGD$, or $BFGE$, is the rectangle required.

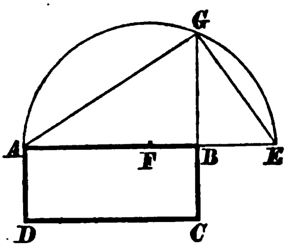


For the triangle ABC, and the *parallelogram* $ABED$, are on the same base AB, and between the same parallels,

∴ the triangle is equal to *half* the parallelogram (41 Cor. 2). Also, since $AF = BF$, the parallelograms $ADGF$, $BEGF$ are equal to one another (41 Cor. 1), that is, each of them is *half* of $ABED$; ∴ each of them is equal to the triangle ABC.

126. Prop. XXIX. *To construct a* square *which shall be equal to a given rectangle.*

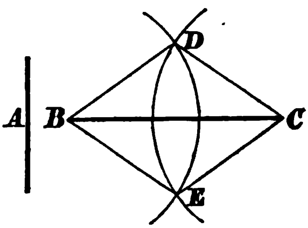
Let $ABCD$ be the given rectangle; produce one side AB to E, making $BE = BC$, the other side of the rectangle: bisect AE in F; with centre F, and radius AF, describe a semi-circle AGE, producing CB to meet the circumference in G. The square of BG is the square required.



For, joining AG, EG, $\angle AGE$ is a right angle, being the 'angle in a semi-circle', ∴ BG is a '*mean proportional*' between AB, and BE, that is, between AB and BC (72 Cor. 2); and ∴ the rectangle AB, BC = the square of BG (74).

127. Prop. XXX. *To construct a* lozenge *with its side equal to a given straight line.*

Let A be the given straight line; take any straight line BC, *less than twice* A; with centres B and C, and radius equal to A, describe intersecting arcs at D and E, on opposite sides of BC. Join BD, BE, CD, CE; and $BDCE$ is the lozenge required.



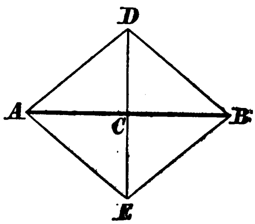
The proof is obvious; and there will be various forms, all satisfying the conditions of the question, according to the *assumed* magnitude of BC.

It is to be remembered, however, that the case is excluded, in which the *diagonals* BC, DE are *equal*, because then the figure $BDCE$ will be a *square*, and not a *lozenge*.

It may also be worth remembering, that besides the common property of *equal sides,* the *square* and *lozenge* have another property in common, viz. that the *diagonals* are in every case *at right angles* to one another, as may easily be proved.

128. Prop. XXXI. *To construct a* lozenge *with given diagonals, that is, with diagonals equal respectively to two given straight lines.*

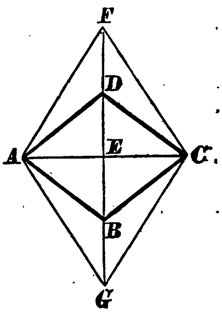
Take AB equal to one of the given lines; bisect AB in C; from C draw CD, and CE, on opposite sides of AB, and at right angles to AB; and make $CD = CE =$ half of the other given straight line. Join AD, AE, BD, BE, and $ADBE$ is the *lozenge* required.



That it is a *lozenge* will easily be shewn by (24); and the *diagonals* are equal to the given lines *by construction.*

129. Prop. XXXII. *To construct a lozenge which shall be double of a given lozenge, and have one diagonal the same.*

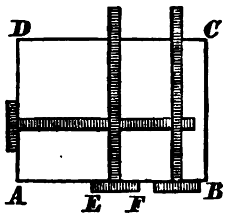
Let $ABCD$ be the given lozenge, and AC the given diagonal. Join BD, meeting AC in E; and produce it both ways to F and G, making $DF = DE = BG$. Join AF, AG, CF, CG, and $AFCG$ is the lozenge required.



For, since the triangles DEA, FDA are upon *equal* bases DE, DF, and 'between the same parallels', they are equal to one another (41 Cor. 2); ∴ the triangle AEF is *double* of the triangle AED. But AEF is *one-fourth* of the lozenge $AFCG$; and AED is one *fourth* of the given lozenge; ∴ $AFCG$ is *double* of $ABCD$.

130. Prop. XXXIII. *To explain the* T *square, and the Drawing-Board.*

(1) The *Drawing-Board* is a smooth board, as *ABCD*, very accurately *rectangular*, with its edges quite smooth. Upon this board drawing-paper is usually fixed, and evenly stretched, by means of glue or paste applied to a small strip of the paper all round, which it is not intended to retain in the drawing.



(2) The T *square* is an instrument consisting of two parts, one called the *stock*, as *EF*, into which the other, called the *blade*, is fixed, near the middle of it, *at right angles* to *EF*. The *blade* is a thin flat-ruler, and the *stock* is somewhat thicker; so that there is a projecting edge of the *stock*, which enables the draughtsman to slide the *square* backwards or forwards along the *drawing-board*, keeping *EF* in close contact with *AB*, while the *blade* continues to lie flat, in every position of the *square*, upon the drawing-paper. Thus, the *blade* being a ruler which is always at right angles to *AB*, it is evident that any number of *parallel* lines may be drawn along it on the paper which is fixed upon the Drawing-Board. And, again, by removing the *square* to one of the adjacent sides of the board, as *AD*, it is also evident that any number of lines may be drawn at pleasure *at right angles* to the former, and parallel to one another.

It is scarcely necessary to point out the advantage of the above combination to *architectural* draughtsmen, or to any others, who are required in the same drawing to trace a *series* of lines *parallel* to each other. Of course, for accurate work both the *Drawing-Board* and the *Square* must be *accurately* constructed. The former may be tested by measuring its opposite sides, *and its diagonals*. For, in a true rectangle *ABCD*, $AB = CD$, $AD = BC$, and $AC = BD$, the whole *three* conditions must be satisfied. Also the T *square* may be tested in the manner pointed out for the ordinary *square* in (96).

131. Prop. XXXIV. *To find the* radius, *and the* centre, *of a given circle.*

Sometimes a circle is said to be *given*, when the *radius* is known, because the *radius* alone is sufficient to fix the *magnitude* of the circle. It is not, however, in such case *fully* given, unless the *position of the centre* be also known. *Here* the circle is supposed to be given by being simply presented before our eyes, without either *centre* or *radius* marked upon it.

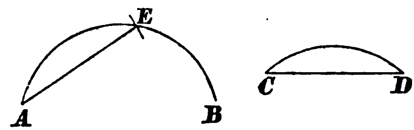
(1) Two methods of finding the *centre* are given in (50), both sufficiently practical. The first method may be applied to any *arc*, or *segment*, of a circle, as well as to a whole *circle*.

(2) Of course, when the *centre* is found, any straight line drawn from it and terminated by the circumference is the *radius*.

(3) If the given circle be such, that no straight lines can be drawn *within* it, as a circular fish-pond or the mouth of a coal-pit, the *radius* may still be found by taking certain *measurements*, as will be shewn in Part III. on *Mensuration.*

132. Prop. XXXV. *From a given arc of a circle to cut off a part equal to another given smaller arc of the same circle, or of a circle with the same radius.*

Let AB be a given arc from which it is required to cut off a part equal to CD, another given arc with the same radius. Join the *chord* CD; then with

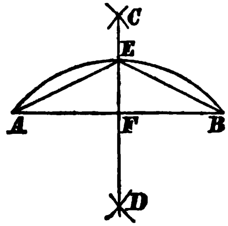


the centre A, and radius equal to the *chord* CD, describe a small arc cutting the arc AB in E; the arc AE is the arc required.

For, drawing the *chord* AB, the *chord* AE = *chord* AB; and equal *chords* in the same circle, or in circles of equal radii, subtend equal *arcs* (58); ∴ *arc* AE = *arc* CD.

133. Prop. XXXVI. *To* bisect *a given arc of a circle.*

Let A and B be the extreme points of the arc which it is required to bisect. With centres A and B, and any convenient radius greater than half the *chord* AB, describe two pairs of intersecting arcs on opposite sides of



the *chord AB*, and let C, D be the *points* of intersection. Join CD, meeting the *arc AB* in E; the *arc AB* is bisected in E.

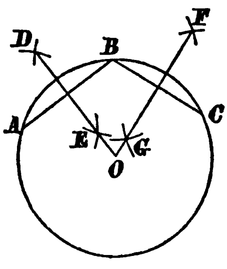
For, joining AB, AE, BE, let CD meet the *chord AB* in F; then, by (101), *CD bisects AB* in F, and is *at right angles* also to it; $\therefore$ the triangles AFE, BFE, are equal in all respects (24); and $\therefore$ the side AE = the side BE. But equal chords in the same circle subtend equal arcs (58); $\therefore$ *arc* AE = *arc* BE.

If the point C fall on the same side of the arc AB as D, then CD being joined, DC must be *produced* to meet the arc in E.

Or, if for want of room, or other cause, the *two* pairs of intersecting arcs cannot be drawn on *opposite* sides of the chord AB, they may be drawn on the same side in the manner pointed out in (101).

134. Prop. XXXVII. *To construct a circle whose circumference shall pass through, 1st,* two *given points, and 2nd,* three *given points.*

(1) Let A and B be two given points. With centres A and B, and any convenient radius describe two pairs of intersecting arcs; join the points of intersection, D, E, and *any* point in DE, or DE produced, being taken for the centre, if a circle be described to pass through A, it will also pass through B.



For *DE bisects AB at right angles* (101), and therefore passes through the centre of any circle of which AB is a *chord* (49).

(2) Let C be a third given point; proceed as before with the two points B, C, drawing FG to bisect the line BC at right angles. Produce FG and DE, if necessary, till they intersect in O. With centre O, and radius OA, describe a circle, and it shall pass through A, B, and C.

For the centre of *every* circle, which can be made to pass through A and B, is in DE, or DE produced; and

the centre of *every* circle, which can be made to pass through B and C, is in FG, or FG produced. And the only point which these lines have in common is their point of intersection O; that is, the only circle which can at the same time pass through A and B, as well as B and C, is that which has O for its centre.

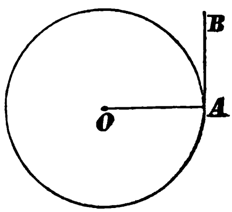
N. B. The only case in which this construction will fail is, when the three given points are in one and the same straight line. In that case the lines DE, FG will be *parallel*, and ∴ will never *meet* in a point O.

COR. 1. *Any number* of circles* can be drawn through *one* given point, or *two;* but no more than *one* distinct circle can be drawn through *three* given points. Hence *three points, given in the circumference of a circle, are sufficient to fix both the magnitude and position of the circle.*

COR. 2. Hence, also, no two distinct circles can cut one another in more than *two* points. For, if they could cut one another in *three* points, then assuming those points for the *three given points* of the Proposition, *two* distinct circles would be drawn through them, which, by Cor. 1, is impossible.

135. PROP. XXXVIII. *To draw a straight line* 'touching' *a given circle, 1st, from a given point* in *the circumference, and 2nd, from a given point* without *it.*

(1) Let O be the centre of the given circle†, and A the given point in the circumference. Join OA, and draw AB *at right angles* to OA; AB *touches* the circle at the point A.



The proof is given in (55).

(2) When the given point is *without* the circumference, apply the method given in (56).

N. B. The application of this Prop. is of constant occurrence in Mechanical Drawing; and requires to be strongly impressed upon the beginner, because it *appears*, at first sight, so *easy* a matter, without any theory, to draw a straight line touching a given circle; whereas, out of the numberless straight lines,

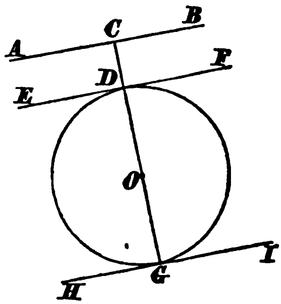
* Here, and in numerous other places, we use the word *circle*, for shortness, when we really mean the *circumference* of the circle (19).

† The centre is either at once *given*, or may be found by (50).

which may be drawn through the given point, only *one* in the 1st case, and *two* in the 2nd case, will really *touch* the circle; and, so far from being *easy*, it is barely *possible*, to *guess* at a true *tangent* of a circle *by the eye*.

136. Prop. XXXIX. *To draw a* tangent *to a given circle which shall also be parallel to a given straight line.*

Let AB be the given straight line; and find O the centre of the given circle. Draw OC perpendicular to AB, meeting AB in C, and the circumference of the circle in D; through D draw EDF parallel to AB, and EDF shall be the line required.

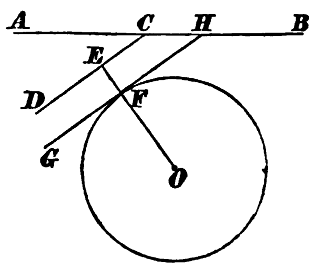


Or, produce DO to meet the circumference in G, and through G draw HGI parallel to AB, or at right angles to DG; then HI is the line required.

For, since $\angle ACO$ is a right angle, and ED is parallel to AC, $\therefore$ $\angle EDO$ is a right angle (34 Cor. 4); and $\therefore$ ED *touches* the circle at D (55). Similarly, HG *touches* the circle at G, and is parallel to AB.

137. Prop. XL. *To draw a* tangent *to a given circle which shall also make with a given straight line an angle equal to a given angle.*

Let AB be the given straight line; take any point C in AB, and from C draw CD making with AB an angle equal to the given angle (105). Find O the centre of the given circle, and draw OE perpendicular to CD, meeting the circumference of the circle in F. Through F draw GFH parallel to CD, or at right angles to OF; then GFH is the tangent required.

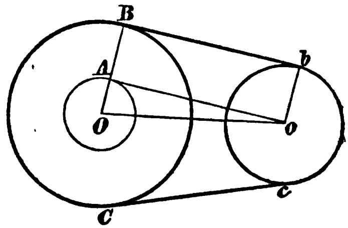


For, it *touches* the circle at F, since $\angle\, OFG$ is a right angle (55), and, since it is parallel to CD, $\angle\, AHG = \angle\, ACD$ (34) = the given angle.

138. Prop. XLI. *To draw a straight line which shall touch each of two given circles.*

There will be two cases of this Prop. 1st, when the *touching* line is wholly on one side of the line joining the centres of the circles; and 2nd, when it *crosses* that line.

Let O, o, be the centres of the two given circles, of which the *greater* has O for its centre, and, for the 1st case, with centre O, and radius equal to the *difference* of the two given radii, describe a circle, and draw from the point o a straight line oA *touching* this circle in the point



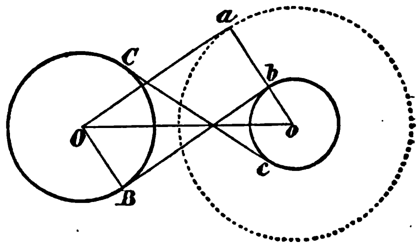
A (56); through A draw OAB meeting the circumference of the larger circle in B; through B draw Bb, parallel to oA, meeting the smaller circle in b; then Bb is the straight line required.

For, since $\angle\, OAo$ is a right angle (56), and Bb is *parallel* to oA, $\therefore$ also $\angle\, OBb$ is a right angle; and $\therefore$ Bb *touches* the larger circle at the point B (55).

Again, joining oB and ob, since Bb is parallel to oA, and $AB = ob$, the triangles ABo, oBb are equal in all respects, and $\therefore$ $\angle\, obB = \angle\, oAB$ = a right angle, $\therefore$ Bb *touches* the smaller circle at b (55).

In the same manner another straight line may be drawn on the opposite side of Oo touching the two given circles, as Cc.

2nd case, when the common tangent is required to *cross* the line joining the centres of the circles.



With centre o, and radius equal to the *sum* of the

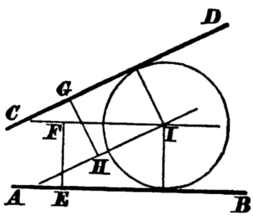
given radii, describe a circle, and draw from O a straight line Oa *touching* this circle in the point a (56). Join oa, intersecting the circumference of the smaller circle in b; through b draw bB, parallel to Oa, meeting the larger circle in B; then Bb is the straight line required.

The proof is similar to that in the first case. Also a *second* tangent may be drawn common to the two given circles, as Cc, by drawing the *second* tangent from O to the circle whose radius is oa.

N. B. These constructions are of great practical value, seeing that they are the exact representations of the modes employed in machinery for communicating motion by means of pulleys and cords, or drums and bands.

139. Prop. XLII. *To draw a circle of given radius* touching *each of two given straight lines.*

Let AB, CD be the two given straight lines. From any points E, and G, in AB and CD draw EF, GH *at right angles* to AB and CD respectively, making $EF = HG =$ the given radius. Through F and H draw FI, HI parallel to AB and CD respectively, intersecting one another in I. Then with centre I, and the given radius, describe a circle, and it shall *touch* both AB and CD.



By drawing from I straight lines parallel to EF, and GH, the proof is obvious (40).

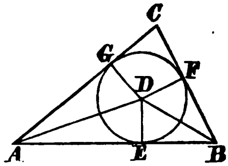
If the given straight lines be *parallel*, the problem is not possible except in the particular case when the given radius is equal to half the perpendicular distance between the parallels.

The given lines may be *at right angles* to each other, and the above construction still holds.

140. Prop. XLIII. *To draw a circle which shall* touch *each of* three *given straight lines.*

(1) Let the three given straight lines be produced until they meet in A, B, and C, forming the triangle

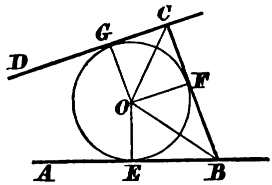
ABC. Bisect the angles *BAC, ABC* by the straight lines *AD, BD* intersecting in *D*. From *D* draw *DE* perpendicular to *AB*; and with centre *D* and radius *DE* describe a circle. This circle shall be the circle required.



For, if *DF, DG* be drawn perpendicular to *BC, AC* respectively, it may easily be shewn that $DE = DF = DG$, and the angles at *E, F*, and *G* are *right angles* by construction; ∴ *AB, BC, AC touch* the circle at the points *E, F, G* (55).

(2) *Another case.* If it be inconvenient to produce the given lines to their points of intersection, so as to form a *triangle*, or if two of them be *parallel*, so that they *cannot* meet, nearly the same method may be applied as follows:—

Let *AB, BC, CD* be the three given straight lines, of which *BC* meets the other two in *B* and *C*, forming the angles *ABC, BCD*. Bisect the angles *ABC, BCD* by the straight lines *BO, CO*, intersecting in *O*. From *O* draw *OE* perpendicular to *AB*. Then with centre *O*, and radius *OE*, describe a circle, and it shall be the circle required.



For, drawing *OF, OG* perpendicular to *BC, CD* respectively, and joining *OB, OC*, it will easily be shewn that $OE = OF = OG$, and the angles at *E, F, G* are *right angles* by construction.

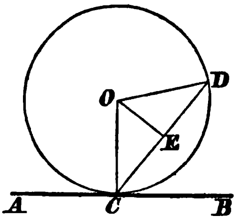
The 1st case is Euclid's Proposition "To inscribe* a circle in a given triangle".

141. PROP. XLIV. *To draw a circle whose circumference shall pass through a given point, and touch a given straight line at a given point in it.*

Let *AB* be the given straight line; *C* the given point

* DEF. A circle is '*inscribed*' in a rectilineal figure (not merely when it is drawn *within* the figure, but) when *each side* of the figure *touches* the circle.

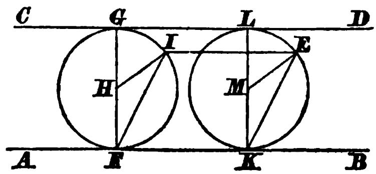
in it; D the other given point through which the circle is to pass. Join CD; bisect it in E; draw EO at right angles to CD, and CO at right angles to AB. With centre O, and radius OC, describe a circle, and it shall be the circle required.



For, since $CE = ED$, and $\angle CEO$ is a right angle, EO must contain the centre of every circle of which CD is a *chord* (49 Cor.). Also, since CO is at right angles to AB, the circle, whose centre is O and radius OC, *touches* AB in the point C (55).

142. Prop. XLV. *To draw a circle whose circumference shall* touch *each of two given parallel straight lines, and pass through a given point between them.*

Let AB, CD be the two given parallel straight lines; and E the given point between them. In AB take any point F, and draw FG at right angles to AB, meeting CD in G. Bisect BG in H. With centre H, and radius HF, describe a circle, which will touch AB, CD in F and G (55). Through E draw EI parallel to AB or CD, meeting the circumference of this circle in I. Join FI; draw EK parallel to FI, meeting AB in K. Through K draw KL at right angles to AB. Bisect KL in M; and with centre M, and radius MK describe a circle. This shall be the circle required.



For, joining HI, ME, it may easily be shewn that $ME = MK$, and $\therefore$ the circle passes through E. Also the angles at K and L with AB and CD are *right angles*; $\therefore$ the circle *touches* AB and CD.

143. Prop. XLVI. *To draw a circle of given radius* touching *another given circle.*

Let A be the centre of the given circle (see figs. 62); draw any radius AC in it, and

1st. If the required circle is to touch the other *externally*, produce AC, until the part CB produced is equal to the given radius of the circle to be drawn. With centre B, and radius BC, describe a circle, and it shall be the circle required.

2nd. If the required circle is to touch the other *internally*, in CA take CB equal to the given radius; then with centre B, and radius BC, describe a circle, and it shall be the circle required.

The proof in each case is obvious.

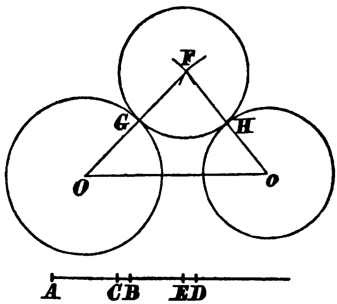
N. B. The point of *contact* between two circles, which *touch* each other, is always in the straight line joining their centres; and the distance between the centres is always either the sum or difference of the radii, according as they touch externally or internally.

Also, at the point of contact the circles have a *common tangent*.

The construction in this problem represents a common case in mechanical drawing, the circles representing two toothed wheels, of which one communicates motion to the other.

144. Prop. XLVII. *To draw a circle of given radius* touching *each of two given circles.*

Let O, o be the centres of the two given circles. Along an indefinite straight line mark off AB, AC equal to the radii of these circles; then mark off BD, CE, each equal to the radius of the *required* circle. With centre O, and radius AD describe a small arc towards the point, as near as you can guess, where the centre of the required circle will lie; and with centre o,



and radius AE, describe another arc intersecting the former arc in F. Join OF, meeting the circumference in G. Then with centre F, and radius FG, describe a circle, and it shall be the circle required.

For, joining oF, meeting the circumference of the

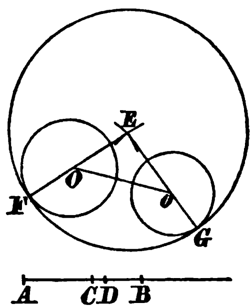
other circle in H, $OF = AD = AB + BD = OG + BD$; $\therefore$ $FG = BD$. Also, $oF = AE = AC + CE = oH + BD$; $\therefore$ $FH = BD$; and $BD =$ the given radius. Also the circles *touch* each other at G and H, by (62).

N. B. This problem requires the centres O, o to be so situated, and the radii of the several circles to be such, that $OF + oF$ shall not be *less than* Oo (38), that is, that the sum of the radii of the two *given* circles, added to the *diameter* of the required circle, be *not less than* the distance between the centres of the given circles. For otherwise the small arcs which determine the point F will not meet at all.

If $OF + oF = Oo$, F will be in the line Oo, and will be found at once by *bisecting* that part of Oo which lies *externally* between the circumferences of the two given circles.

145. Prop. XLVIII. *To draw a circle of given radius so as to be* touched internally *by two other smaller given circles.*

Let the straight line AB be equal to the given radius of the *required* circle; and let the two given smaller circles be those in the annexed fig. with centres O, o. Mark off BC equal to the radius of the former, and BD equal to the radius of the latter. With centre O, and radius AC, describe a small arc, as near as you can guess to the centre of the required circle, and with centre o, and radius AD, describe another arc intersecting the former arc in the point E. Join EO, and produce it to meet the circumference again in F. Then with centre E, and radius EF, describe a circle, and it shall be the circle required.



For, $EF = EO + OF = AC + BC = AB$; and similarly, $EG = AB$; also the circles touch one another at F and G, by (62).

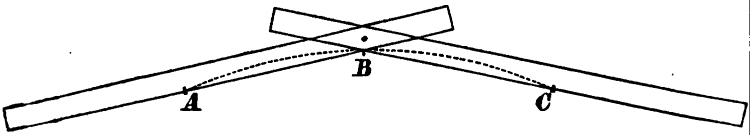
N.B. A similar restriction to that in (144) belongs to this Problem also, viz. $OE + oE$ must not *be less than* Oo; that is, the sum of the radii of the two given circles,

subtracted from the *diameter* of the required circle, must not be *less than* the distance between the centres of the given circles.

146. Prop. XLIX. *To draw an arc of a circle passing through three given points (not in the same straight line), without finding or using the* centre *of the circle.*

[This is required to be done in drawing the parallels of latitude for maps, and also in laying down railway-curves; in both which cases, as well as in some others, the *centre* is often inconveniently remote.]

Let A, B, C be the three given points, of which B



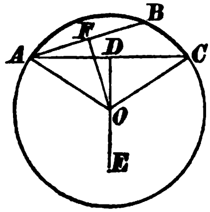
lies between the other two. Through A and C fix two pins, pegs, or nails, in the plane surface on which the arc is to be drawn. Take two '*straight-edges*' or '*flat-rulers*', and lay them flat on the surface with the edge of one resting on A, and of the other on C; and bring them together, until the same straight edges, which pass through A and C, intersect in the point B. When that is the case, *fasten* the rulers tightly together at their junction, so that *afterwards*, during the operation, the *angle* between them cannot vary. Then place a marker at B, in contact with the surface, and while this marker retains a fixed position with respect to the instrument, *slide* the whole instrument on the pins at A and C, so as to bring the junction of the straight edges, which was at B, first to A and then to C, and the marker during this operation will trace out the *arc* required.

For the curved line, whatever it be, certainly passes through A, B, and C: and, if AC be joined, AC subtends the same angle at *every point* in the curved line, which is a well-known property of a segment of a *circle* (52 Cor.).

There is an instrument called a *Bevel* commonly used for the above purpose by those who have frequent occasion to draw such arcs.

147. Prop. L. *An arc, or a segment, of a circle being given, to complete the circle of which it is a part.*

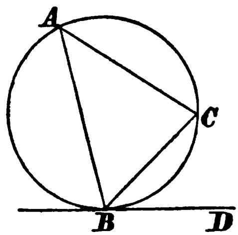
Let ABC be the given arc, or segment; if the former, join A and C, the extreme points, by the *chord* AC. In either case bisect AC in D by the straight line DE at right angles to AC (101). Join AB, B being *any* point in the given arc, and bisect AB in like manner by the straight line FO at right angles to AB, meeting DE in O. Then with centre O, and radius OA, describe a circle, and it will be the circle required.



For, it may easily be shewn, that $OA = OC = OB$, and B is *any* point in the given arc; ∴ the *whole arc* is a part of the circumference thus drawn.

148. Prop. LI. *From a given circle to cut off a segment, which shall contain an angle equal to a given angle*.*

Let ABC be the given circle, A and B being *any* points in its circumference; through B draw BD *touching* the circle (135); and draw the chord BC such that $\angle DBC =$ the given angle (105). Then BAC is the required segment.



For, joining AB, AC, $\angle BAC$ in the alternate segment (63) $=$ $\angle DBC =$ the given angle.

Cor. If the given angle be a *right* angle, the segment will be a *semi-circle*, which of course, may be cut off the given circle by drawing any diameter of the circle.

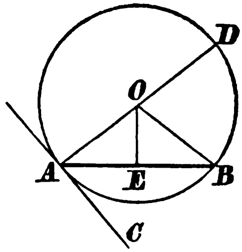
If the given angle be *acute*, the segment will be greater than a semi-circle.

And, if the given angle be *obtuse*, the segment will be less than a semi-circle (54).

* The learner must bear in mind the Definition of '*angle in a segment*'; see (52 Cor.).

149. PROP. LII. *Upon a given straight line* to construct a* segment of a circle *which shall contain an angle equal to a given angle.*

Let AB be the given straight line; and through the point A draw the straight line AC making with AB the angle BAC equal to the given angle (105). From A draw AD at right angles to AC. Bisect AB in E, by the straight line EO (101), intersecting AD in O. With centre O and radius OA describe a circle, cutting AO produced in D; then ADB is the segment required.



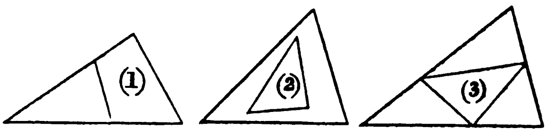
For, since AC is *at right angles* to OA the radius, $\therefore$ AC *touches* the circle at A (55). And the 'angle in the segment' $ADB = \angle BAC$ (63) = the given angle.

If the given angle be a *right angle*, it will then only be necessary to describe a *semi-circle* on AB as a *diameter*, and that semi-circle will be the segment required.

INSCRIBED AND CIRCUMSCRIBED FIGURES, AND CONSTRUCTION OF POLYGONS.

150. DEFINITION 1. A RECTILINEAL FIGURE is said to be '*inscribed*' in another rectilineal figure, when all the *angular points* of the inscribed figure are upon the *sides* of the figure in which it is inscribed, *each upon each.*

Thus, one triangle is '*inscribed*' in another triangle, not merely when the one is *situated within* the other, as in the first or second of the annexed diagrams, but when



* A *segment* of a circle is defined (48) to be a portion of the circle bounded by an *arc* and its *chord.* The *chord* is sometimes called the *base* of the segment, that is, it is a straight line *upon which the segment is supposed to stand.*

also *each* side of the outer triangle has upon it the *vertex* of *each* one of the angles of the inner triangle, as in the 3rd diagram.

Def. 2. A Rectilineal Figure is said to be '*circumscribed*' about another rectilineal figure, when all the *sides* of the circumscribed figure pass through the *angular points* of the figure about which it is circumscribed, *each through each*.

Thus, in the preceding diagrams, the larger triangle is *not* '*circumscribed*' about the lesser in the 1st and 2nd, but only in the 3rd.

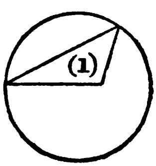
Def. 3. A Rectilineal Figure is said to be '*inscribed in a circle*', when all the *angular points* of the inscribed figure are upon the *circumference* of the circle.

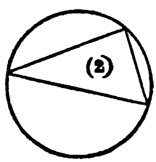
Def. 4. A Rectilineal Figure is said to be '*circumscribed about a given circle*', when *each side* of the circumscribed figure *touches* the circle.

Def. 5. A Circle is said to be '*inscribed*' in a rectilineal figure, when it is so drawn as to *touch* each side of the figure.

Def. 6. A Circle is said to be '*circumscribed*' about a rectilineal figure, when its *circumference* passes through all the *angular points* of the figure.

It is important for the learner to take good heed to these *Definitions*, because the words '*inscribed*' and '*circumscribed*' are allowed to have only the technical meanings here assigned to them, whereas the tendency is to give them a much wider meaning, which leads to serious error. For instance, the *careless* student would say, that the triangle in the annexed fig. (1), is *inscribed in* the circle, or the circle *circumscribed* about the triangle; but it is not so, *according to the Definition*, which requires that *each angle* of the *inscribed* figure have its *vertex* in the *circumference* of the circle, as shewn in fig. (2).

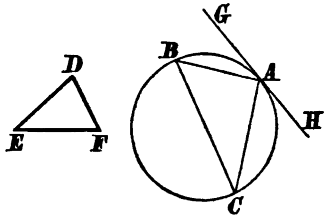




In fact, to call the circle in fig. (1) the *circumscribing* circle of the triangle would be to *define* nothing, because there are *an infinite number* of such circles, having the common property of passing through *two* of the vertices of the triangle; but when we speak of the circle *circumscribing* the triangle *according to Definition*, as shewn in fig. (2), we speak of a particular well-defined circle, for there is *one such* circle, and *one only* (134 Cor. 1).

151. Prop. LIII. *In a given circle to* inscribe *a triangle similar, that is, equiangular, to a given triangle.*

Let ABC be the given circle; and DEF the given triangle. Draw the straight line GAH *touching* the circle in the point A; and from A draw the *chords* AB, AC, such that $\angle HAC = \angle DEF$, and $\angle GAB = \angle DFE$ (105). Join BC; and ABC is the triangle required.



That ABC is a triangle *inscribed* in the circle is plain, because it has the vertex of each of its angles on the circumference. Also, by (63), $\angle ABC = \angle HAC = \angle DEF$; and $\angle ACB = \angle GAB = \angle DFE$; $\therefore$ the remaining $\angle BAC = \angle EDF$ (37); that is, the triangle ABC is equiangular, and $\therefore$ similar, to the triangle DEF.

Obs. Since the point A was taken arbitrarily any where in the circumference, there may be *any number* of such triangles, as *ABC, inscribed* in the circle. But they will all be equal and similar to one another, being different only in *position*.

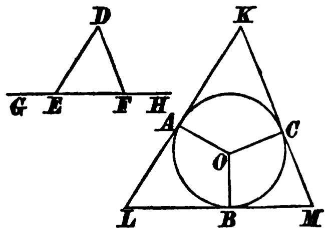
It is also to be noted, that in the above construction we have the solution of the following Problem:—

'*Out of a given circle to cut the* greatest *triangle similar to a given triangle*'.

152. Prop. LIV. *About a given circle to* circumscribe *a triangle similar, that is, equiangular, to a given triangle.*

Let ABC be the given circle; and DEF the given triangle. Produce EF both ways indefinitely to points G and H; find the centre O of the given circle, and draw

in it any radius OA; draw two other radii OB and OC, such that $\angle AOB = \angle DEG$, and $\angle BOC = \angle DFH$ (105). Then through the points A, B, C, draw the straight lines KL, LM, MK, *touching* the circle, and KLM is the triangle required.



For, that the triangle KLM is *circumscribed* about the circle is plain from the construction, because each of its sides is *made* to *touch* the circle. It is also similar to DEF; for, in the quadrilateral $AOBL$ the angles at A and B are both right angles, but *all* the angles of a quadrilateral figure are together equal to *four* right angles, $\therefore \angle AOB + \angle ALB =$ *two* right angles $= \angle DEG + \angle DEF$ (30). But $\angle AOB = \angle DEG$, $\therefore \angle ALB$, or $\angle KLM = \angle DEF$. Similarly $\angle KML = \angle DFE$, and $\therefore \angle LKM = \angle EDF$ (37), that is, the triangle KLM is equiangular to the triangle DEF.

153. Prop. LV. *In a given triangle to* inscribe *a circle.*

Let ABC be the given triangle; see fig. (140.) Bisect the angles BAC, ABC by the straight lines AD, BD, intersecting in D. From D draw DE perpendicular to AB. With centre D, and radius DE, describe a circle, and it shall be the circle required.

For, drawing DF perpendicular to BC, and DG perpendicular to AC, it is easily shewn that $DE = DF = DG$; $\therefore$ the circle described passes through E, F, and G. And the angles at those points are *right* angles, $\therefore AB$, BC, and AC *touch* the circle.

This construction contains the solution of the following Problem:—

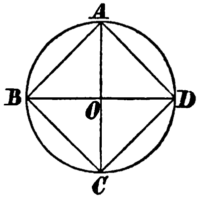
'*To cut the* greatest *circle out of a given triangle*'.

154. Prop. LVI. *About a given triangle to* circumscribe *a circle.*

This is the same construction as that which occurs in (134) where we construct a circle to pass through *three* given points. Here the given points are the vertices of the three angles of the given triangle.

155. PROP. LVII. *In a given circle to* inscribe *a square.*

Let $ABCD$ be the given circle; O the centre; draw two diameters AC, BD *at right angles* to each other. Join AB, AD, CB, CD, and $ABCD$ is the *square* required.



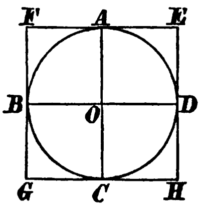
For, since AC, BD are *diameters* of the circle, each of the angles of the figure $ABCD$ is the 'angle in a semicircle', and is ∴ a *right* angle (54). Also AC, BD divide the circumference into four *equal* parts, ∴ AB, AD, CB, CD are *chords* of equal arcs, and ∴ are equal to one another.

COR. If each of the *arcs* AB, BC, CD, DA be *bisected* (133), and *chords* be drawn from the corners of the square to each point of division, a regular *octagon* will be inscribed in the circle.

This construction shews us how '*to cut the* greatest *square, or octagon, out of a given circle*'.

156. PROP. LVIII. *About a given circle to* circumscribe *a square.*

Let $ABCD$ be the given *circle;* O the centre; draw two diameters AC, BD at *right angles* to each other; through the points A, B, C, D draw FE, FG, GH, HE *touching* the circle; $EFGH$ is the *square* required.



For, by (55) the angles at A, B, C, D are right angles; also, by construction the angles at O are right angles; and all the angles of any four-sided figure are *together* equal to four right angles: ∴ each of the angles at E, F, G, H is a *right* angle. Again, by (34 Cor. 2) FE is parallel to BD, and to GH; also FG is parallel to AC, and to EH; ∴ $EFGH$ is a *parallelogram*, ∴ $EF = GH = BD$ (40); and $FG = EH = AC$; but $AC = BD$, ∴ $EF = FG = GH = HE$; and ∴ $EFGH$ is a *square*.

157. Prop. LIX. *In a given square to* inscribe *a circle.*

See fig. in last Prop. Let $EFGH$ be the *given* square. Bisect each of the sides EF, FG in A and B; through A draw AC parallel to FG; and through B draw BD parallel to EF, intersecting AC in O; then with centre O, and radius OA, describe a circle, and it shall be the circle required.

For, since each of the foursided figures are, by construction, *parallelograms*, it may easily be shewn that $OA = OB = OC = OD$, and that the angles at A, B, C, D are *right* angles. Then it follows, that the circle *touches each* of the sides of $EFGH$, and is ∴ '*inscribed*' in it.

This construction shews us how '*to cut the* greatest *circle out of a square*'.

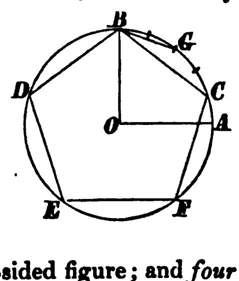
158. Prop. LX. *About a given square to* circumscribe *a circle.*

See fig. in (155). Let $ABCD$ be the given square. Draw the diagonals AC, BD, intersecting in O. With centre O, and radius OA describe a circle, and it shall be the circle required.

For, by (40 Cor. 2) AC, BD *bisect* each other in O; ∴ since $AC = BD$, $OA = OC = OB = OD$, and ∴ the circle with centre O, and radius OA, will pass through B, C, D, as well as A.

159. Prop. LXI. *In a given circle to* inscribe *a regular pentagon.*

Let O be the centre of the given circle; draw OA any radius, and OB another radius *at right angles* to OA, so that AB is an arc of a *quadrant* of the given circle. Divide the arc AB into *five* equal parts*; and let AC be the first of such parts, reckoned from A. Join BC; then draw the chords BD, CF, FE, each equal to BC; and *join* ED. $BDEFC$ shall be the pentagon required.



For, *by construction* it is a *five*-sided figure; and *four* of the sides are *made* equal to one another. Also, ED, the remaining side, is the chord of the arc, which remains after taking from the whole circumference *four-fifths* of

* This may be done *by trial* with the compasses; see (169).

the arc of a quadrant *four* times; for BC is *four-fifths* of AB, by construction; that is, the *arc* ED = the difference between four quadrants and $3\frac{1}{5}$ quadrants = *four-fifths* of a quadrant = *arc* BC; $\therefore$ $BDEFC$ is *equilateral*. It is also *equiangular*, since each angle is the angle in one of five segments all of which are *equal* to one another.

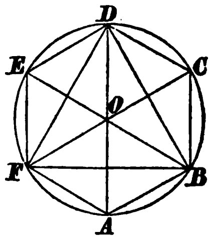
The accuracy of the work may be *tested* by observing whether the *diagonals* DF, DC, BE, BF are all *equal* to one another, as they ought to be.

Cor. If G be the *third* point of division from A, that is, the *second* from B, by joining BG and continuing equal chords round the circle, we *inscribe* a regular *decagon* in the circle.

This construction shews us how '*to cut the* greatest *regular pentagon or decagon out of a given circle*'.

160. Prop. LXII. *In a given circle to* inscribe *a regular hexagon.*

Let O be the centre of the given circle; draw OA any radius; and beginning from A, draw AB, BC, CD, DE, EF five consecutive chords each equal to OA; and *join* FA. $ABCDEF$ is the hexagon required.



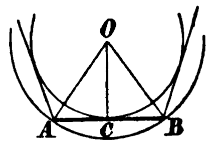
For, *by construction* it is a *six-sided* figure, and *five* of its sides are *made* equal to one another. Also, joining OB, OC, OD, OE, OF, it is plain that each of the triangles OAB, OBC, OCD, ODE, OEF is *equilateral*, and $\therefore$ each of the five angles at O is *one-third* of two right angles (37), that is, *one-sixth* of *four* right angles; and $\therefore$ the sum of them is *five-sixths* of four right angles. But $\angle AOF$ makes up the whole four right angles about the common vertex O (30 Cor. 2), $\therefore$ $\angle AOF$ = *one-sixth* of four right angles; and $\therefore$ the chord AF = each of the other chords; and $ABCDEF$ is *equilateral*. Again, it is *equiangular*, because all the angles are angles in equal segments of the same circle.

Cor. If BD, BF, DF be joined, BDF is an *equilateral triangle* '*inscribed*' in the circle.

This construction shews us how '*to cut the* greatest *regular hexagon, or equilateral triangle, out of a given circle*'.

161. PROP. LXIII. *In a given regular polygon of any number of sides to* inscribe *a circle.*

Let AB be a side of the given polygon; and bisect each of its angles at A and B by the straight lines AO, BO, intersecting in O. From O draw OC, perpendicular to AB; with centre O, and radius OC, describe a circle, and it shall be the circle required.



For the proof of this, see (87 Cor. 3).

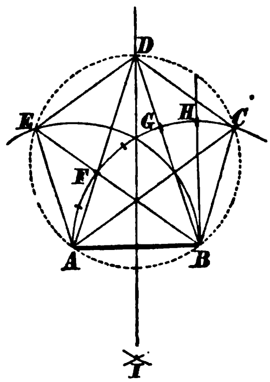
162. PROP. LXIV. *About a given regular polygon of any number of sides to* circumscribe *a circle.*

Let AB be a side of the given polygon, as in (161); and bisect each of its angles at A and B by the straight lines AO, BO, intersecting in O. With centre O, and radius OA describe a circle, and it shall be the circle required.

For the proof of this, see (87 Cor. 1).

163. PROP. LXV. *To construct a* regular pentagon *with each of its sides equal to a given line or length.*

Let AB be the given straight line or length; with centres A and B, and radius AB, describe two arcs, somewhat greater than those of quadrants, on that side of AB on which the pentagon is to lie, and two small arcs on the other side intersecting in the point I. Draw the indefinite line ID, through the intersections of these arcs; and through B draw BH parallel to ID, and meeting the circumference of one of the circles in H, so that ABH is a *quadrant*. Divide the arc AH into *five* equal parts, marking the second division F, and the fourth G, reckoning from A. Make the arc HC, on the other side of H, equal to the arc HG (132). Join BC, BG, BF; produce BF to meet

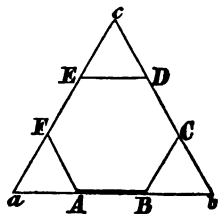


the arc, whose centre is A, in E: and produce BG to meet ID in D. Then join CD, DE, EA; and $ABCDE$ is the pentagon required.

For, joining AC, AD, BE, each of the angles AEB, ADB, ACB may easily be shewn to be *two-fifths* of a right angle; $\therefore$ these angles being on the same base AB, and *equal* to one another, they are '*angles in the same segment*', that is, A, B, C, D, E are points in the circumference of the same circle. Supposing this circle drawn, since $\angle EBD$ = *two-fifths* of a right angle $= \angle CBD = \angle ADB$, and since equal angles, whether *at* the centre or circumference, in the same circle are subtended by equal chords, $\therefore ED = CD = AB$; and $BC = AB = AE$; $\therefore ABCDE$ is *equilateral*. And it is also equiangular, for $\angle ABC$ was made equal to six-fifths of a right angle (86 Cor. 1), and each of the other angles will easily be shewn to be equal to this.

164. Prop. LXVI. *To construct a* regular hexagon *with each of its sides equal to a given line or length.*

Let AB be the given line or length; produce it both ways to a, and b, making $Aa = Bb = AB$. Upon ab describe an equilateral triangle (23), on that side of it on which the hexagon is to lie; and divide each of the sides bc, ac, into three equal parts in the points C, D, and E, F. Join AF, BC, DE, and $ABCDEF$ shall be the hexagon required.

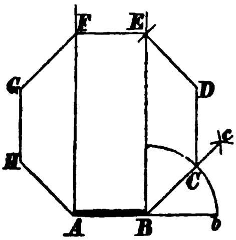


For the triangles AFa, BCb, DEc being equilateral and equiangular, and equal to one another in all respects, the proof is obvious.

This construction enables us to lay down a regular hexagon *on the ground* with remarkable ease and certainty. For it may be done evidently with any long staff, or tape, or chord, *only*, on which the given length of the side is marked.

165. Prop. LXVII. *To construct a* regular octagon *with each of its sides equal to a given line or length.*

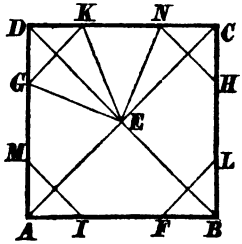
Let AB be the given line or length; produce it indefinitely to b; from A and B draw AF, BE at right angles to AB, on that side of it on which the octagon is required. Bisect $\angle EBb$ by the straight line BC, making BC equal to AB. (If the compasses from the first be set to AB, they will need no change through the whole operation; for the angle EBb may be bisected with that radius as well as any other.) Through C draw CD parallel to BE, and equal to AB; with centre D describe an arc cutting BE in E; through E draw EF parallel to AB; through F draw FG parallel to BC, and equal to AB; through G draw GH parallel to AF, and equal to AB; and join AH. $ABCDEFGH$ is the octagon required.



For all the sides, except AH, are *made* equal to one another; and by joining HF, and BD, the triangles AFH, BED may easily be shewn to be equal in all respects (24); and $\therefore AH = DE = AB$. $\therefore ABCDEFGH$, which has *eight* sides, is *equilateral*. It is also *equi-angular;* for $\angle ABC$, by construction, is equal to one right angle and a half. Also, $\angle DCc = \angle EBC$ (34) = half a right angle, $\therefore \angle BCD$ = one right angle and a half (30) $= \angle ABC$. And the same may be proved of the other angles. (See 86 Cor. 1).

166. Prop. LXVIII. *To cut the greatest regular octagon out of a given square.*

Let $ABCD$ be the given square. Draw its diagonals AC, BD intersecting in E. With centre A and radius AE, mark off AF in AB; and AG in AD; and do the same from each of the other corners, so that $AE = AF = AG = BH = BI = CK = CL = DM = DN$. Join FL, HN, KG, MI, and $IFLHNKGM$ shall be the octagon required.



For, since the sides of the square are equal, it is obvious

enough from the *construction*, that $IF = HL = KN = GM$. But it is not so obvious that the *alternate* sides are equal to these and to one another, for instance that $GK = KN$. To prove this, join EG, EK, EN. Then, since $AG = AE$, $\angle AGE = \angle AEG$ (26); and $\angle EAG =$ half a right angle, $\therefore$ $\angle AGE + \angle AEG =$ one right angle and a half (37), and $\therefore$ $\angle AGE =$ *three-fourths* of a right angle. But $DG = DK$, $\therefore$ $\angle DGK =$ half a right angle, $\therefore$ $\angle EGK =$ *three-fourths* of a right angle. In the same manner it may be shewn, that $\angle EKG = \angle EKN =$ *three-fourths* of a right angle $= \angle ENK$; $\therefore$ the two triangles EGK, EKN have two angles in one equal to two angles in the other, each to each, and one side common, $\therefore$ the triangles are equal, and the sides are equal which are opposite to equal angles (39), that is, $GK = KN$. The same may be proved for any other two adjacent sides of the octagon; $\therefore$ it is *equilateral*. It is also *equi-angular*, since *each* of its angles is obviously equal to one right angle and a half.

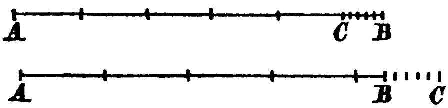
167. Obs. Constructions have now been given for regular figures of 3, 4, 5, 6, 8, and 10, sides. Of course, by *bisecting* the arcs of the circumscribing circle subtended by these sides in any case, a regular polygon of *double* the number of sides is obtained, that is, we can add to the above other polygons of 12, 16, and 20 sides; and again, by subdividing, we have polygons of 24, 32, and 40 sides; and so on. It is to be observed, that, while we do this, we at the same time divide *the circumference of a circle* into as many equal parts as the polygon has sides; and thus the whole circumference of any circle may be minutely and equally subdivided in a great variety of ways. But it is not true, that we can in this way divide the circumference of a circle into *any proposed number* of equal parts, because in attempting this we shall often find ourselves brought to the necessity of *trisecting an arc*, a problem for which no strictly geometrical solution has yet been discovered. We can *trisect* some *particular* arcs, as the arc of a quadrant, of a semicircle, and of a whole circle, but not of *any* arc; and so we are hindered from effecting with theoretical exactness that minute subdivision of the circle which is in common use for scientific purposes. The consequence is that, for most practical purposes, both whole circumferences of

given circles, and given arcs are divided into equal parts *by trial*, either with the ordinary compasses alone, as in (169), or with the help of another circle which is itself already graduated, called a *Protractor*. Such an instrument, and its use, will be explained in Part III.

PROPORTIONAL LINES AND AREAS.

168. Prop. LXIX. *To divide a given straight line into any proposed number of equal parts.*

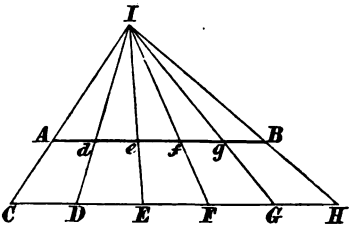
(1) This may be done, in most cases, *by trial* with the compasses alone without any appreciable error. Thus let AB be the given line, which is required to be divided, suppose, into 5 equal parts. Make a guess at the 5th part



of AB; take this distance in the compasses, and step along AB from A to B. If in the 5th step the foot of the compasses falls exactly upon B, then, by marking each step on the line, the thing required is done.—But it is more probable, that the given line will not be thus equally subdivided *at the first trial*. The 5th step of the compasses will be more likely either to fall short, or pass beyond, the point B, by the short length BC. Then, if this short length be divided into 5 equal parts *by the eye*, (which may be done with sufficient accuracy for most practical purposes), and one of such parts be added to the distance in the compasses, or subtracted from it, as the case may be, then with the distance so corrected step along AB, and it will divide AB into the required number of equal parts.

(2) The theoretically exact method of doing the same thing is given in (68); and another method of a similar kind is as follows:—Let AB be the the given straight line which it is required to divide into 5 equal parts. From A draw AI making any *acute* angle with AB; produce IA to any point C; through C draw an indefinite straight line *parallel* to AB; and taking a

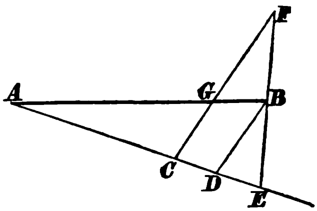
distance in the compasses, at a guess, somewhat greater than the 5th part of AB, step along this line, beginning from C, marking the points of division so that $CD = DE = EF = FG = GH$. Join HB, and produce it to meet CI in the point I. Then join ID, IE, IF, IG, intersecting AB in the points d, e, f, g; and AB shall be divided by these points as required.



For, since ICD, IAd are *similar* triangles, by (71) $Ad : CD :: Id : ID$. Similarly $de : DE :: Id : ID$, $\therefore$ $Ad : CD :: de : DE$; and, alternately, $Ad : de :: CD : DE$ (74 Cor. 3). But $CD = DE$, $\therefore$ $Ad = de$. In the same manner it may be shewn, that $de = ef = fg = gB$.

(3) Another still more ingenious method, which avoids entirely the drawing of any parallel line, is as follows:

From one extremity A of the given line draw an indefinite straight line, making any acute angle with AB; along which make *six* equal steps with the compasses, marking the 4th, 5th, and 6th, by the letters C, D, E. Join EB, and produce it to F, making $BF = EB$. Then join FC, intersecting AB in G, and BG shall be the 5th part of AB; so that taking this distance, BG, in the compasses and stepping along AB from either A or B, the given line is divided as required.



For, joining BD, since $ED = DC$, and $EB = BF$, $\therefore$ the sides EC, EF of the triangle ECF are divided *proportionally* in the points D and B; and $\therefore$ BD is *parallel* to CF. Again, in the triangle ABD, since GC is *parallel* to BD, $AG : AB :: AC : AD$ (70 Cor.); but

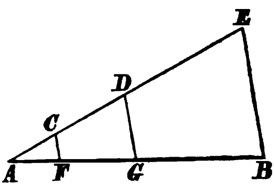
AC is *made four-fifths* of AD, $\therefore$ AG is *four-fifths* of AB; and $\therefore$ BG is *one-fifth* of AB.

169. Prop. LXX. *To divide a given* arc of a circle *into any number of equal parts.*

Since equal chords in the same circle subtend equal arcs (58), this may be done *by trial*, with the compasses, in the same manner as it was done for a straight line by the 1st method in (168); and, as was stated in (167), there is no geometrical method *theoretically exact*, and applicable in all cases.

Obs. In all cases of subdividing a line, whether curved or straight, *by trial*, the operation is performed with the greatest exactness and the least trouble by using a particular form of compasses, called *Hair-Compasses*, or *Hair-Dividers.* This instrument differs from the ordinary compasses only in having a fine screw so connected with *the lower half* of one leg, that, by turning the screw, the foot of that leg is moved through a *very small* distance without disturbing the angle at which the legs are previously set. By this contrivance, when the compasses have been opened so as nearly to embrace any required distance, we can bring the points nearer to, or farther from, each other at pleasure by the smallest possible amount, with an ease and nicety, which is not attainable by moving the *hinge* of the compasses only.

170. Prop. LXXI. *To divide a given straight line into any number of parts which shall be to each other in given ratios.*



Let AB be the given straight line; and, as the process is the same whatever be the numbers, let it be required to divide AB into 3 parts in the ratio of 2, 3, 5.

From A draw an indefinite straight line making any acute angle with AB. With some small opening of the compasses set off $2+3+5$, that is, 10 equal distances along this line, beginning from A. Mark the 2nd point of division C; the 5th D; and the 10th E. Join EB;

and draw CF, DG both parallel to EB, meeting AB in F and G. Then AB is divided in the points F and G as required.

For, by (69) $AF : FG :: AC : CD :: 2 : 3$,

and ... $FG : GB :: CD : DE :: 3 : 5$;

$\therefore$ AF, FG, GB are in the ratio of 2, 3, 5.

Or, if the ratios be given by straight lines, from A set off AC equal to the 1st of them, CD equal to the 2nd, and DE equal to the 3rd. Join EB; and draw CF, DG each parallel to EB. Then AB is divided as required in the points F and G.

Or, again, if the problem be to divide AB into the same number of parts, and having the same ratio to each other, as those into which another given line is divided, the process is obviously still very simple. It is only necessary to place the given line in the position AE, to join BE, and through the *given* points of division, C, D, &c. to draw lines parallel to BE, meeting AB in F, G, &c.

N.B. In copying plans and drawings it often becomes necessary to *transfer* a series of different lengths from one straight line to another, and, of course, this may be done with the *compasses*—but, in practice, the following method is to be preferred. Take a strip of paper, with one edge cut accurately straight, and of sufficient length, and lay its straight edge along the divided line; mark upon it with a finely pointed pencil the exact points which coincide with the points of division, and then lay the same edge along the other line to be divided, and it is evident that the required points of division may at once be marked upon it.

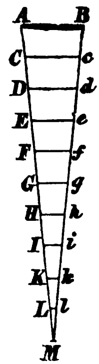
171. Prop. LXXII. *From a given straight line to cut off any proposed part, as one-fifth, one-tenth, &c.*

Let AB be the given straight line, see fig. (170); and through A draw a straight line making any acute angle with AB. In this line take a point C, not far from A, and make AE, on the same line, the same multiple of AC, that AB is of the part to be cut off from it; (that is, if the part required to be cut off from AB is *one-tenth* of it, make AE equal to *ten times* AC). Join EB; and draw CF parallel to EB. Then AF is the part required.

For, by (70 Cor.) $AF : AB :: AC : AE$, that is, AF is the same part of AB that AC is of AE. Thus, if AC be *made one-tenth* of AE, then AF is *one-tenth* of AB.

172. PROP. LXXIII. *To find any proposed part of a given* short *straight line.*

This Problem might be supposed to be included in (168); but practically it is not so, when the given line is a *very short* one. For neither can such a line be readily divided into a great number of equal parts *by trial with the compasses,* nor can a line be very correctly drawn *parallel* to it*, as required by the 2nd method. But the following method may be applied, however small the given line may be. The process being the same in all cases, let it be required to find the *tenth* part of the very short line AB.



From A draw any indefinite straight line making any angle with AB; and with any convenient opening of the compasses make ten successive steps, AC, CD, DE, EF, FG, GH, HI, IK, KL, LM. Join BM, and divide it into ten equal parts by the points c, d, e, f, g, h, i, k, l; and join Cc, Dd, Ee, Ff, Gg, Hh, Ii, Kk, Ll. Then Ll shall be equal to the *tenth* part of AB.

For, since ML is the tenth part of AM, and Ml is the *tenth* part of BM, $\therefore$ the sides AM, BM of the triangle AMB are cut *proportionally* by the straight line Ll, and $\therefore$ AMB, LMl are *similar* triangles (71); and $\therefore$ $Ll : AB :: ML : MA :: 1 : 10$.

COR. Not only do we thus determine the *tenth* part of AB, but Kk = *two-tenths*, Ii = *three-tenths*, Hh = *four-tenths*, Gg = *five-tenths*, Ff = *six-tenths*, Ee = *seven-tenths*, Dd = *eight-tenths*, and Cc = *nine-tenths*, of AB.

It is upon this principle that the '*Diagonal Scale*', so much used in *Mensuration,* is constructed. The construction and use of that *Scale* will be explained in Part III.

* Although it is one of our *Postulates* (20), that is, a truth *to be granted* without proof, that "a terminated straight line may be *produced*, that is, extended, to any length in a straight line", yet *in practice* a *very short* line cannot be '*produced*' with any *certain exactness* by means of the *flat ruler*, or *straight edge* (a fact, which the draughtsman ought constantly to bear in mind); and on that account it is not an easy matter, when a given straight line is *very short*, to draw one or more other straight lines *correctly parallel* to it.

173. Prop. LXXIV. *To find a* fourth proportional *to three given straight lines.*

No more practical method of doing this can be given, than that which is found in (77).

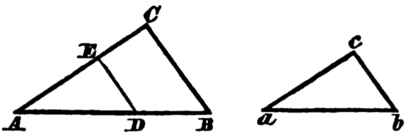
Also, '*to find a* third proportional *to two given straight lines,*' see Cor. (77).

Again, '*to find a* mean proportional *between two given straight lines,*' see Cor. 2 (72).

These constructions are of continual occurrence.

174. Prop. LXXV. *Upon a given straight line as a base to construct a triangle* similar *to a given triangle.*

Let ABC be the given triangle, and ab the given



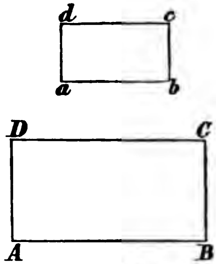
line, which is to be the base of the required triangle. From a draw ac making $\angle bac = \angle BAC$; and from b draw bc making $\angle abc = \angle ABC$ (105). Then abc is the triangle required.

For *two* angles of the triangle abc being *made* equal to *two* angles of ABC, the remaining $\angle acb = \angle ACB$ (37); $\therefore$ the triangle abc is equiangular, and $\therefore$ similar, to the triangle ABC.

Or, if the straight line which is to be the base, be given in *magnitude* only, and not in *position*, set off on AB the line AD equal to it, and through D draw DE parallel to BC. Then ADE is the triangle required (71 Cor. 2).

This is the same Problem as that which requires us '*Having a given triangle, to draw the same on a different scale*'; and it is much used in that part of a *Surveyor's* business, called '*plotting*', which consists in laying down on paper, *on a small scale,* the large triangles which he has actually measured on the Earth's surface. This will be further explained in Part III. on *Mensuration.*

175. Prop. LXXVI. *Upon a given straight line as a base to construct a rectangle* similar *to a given rectangle.*



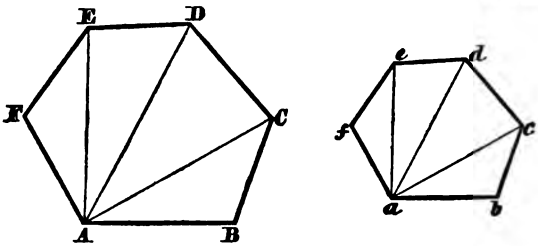
[Although it is true, as stated in (78), that *all* squares are *similar*, and that triangles which are equiangular are necessarily *similar*, it is not true that rectangles, though equiangular, are necessarily *similar;* for this equality of *angles* may obviously exist without the sides being *proportional.*]

Let $ABCD$ be the given rectangle, and ab the given base for the required rectangle. Draw ad, bc, at right angles to ab; find a fourth proportional to AB, AD, ab (77); make ad, bc each equal to it, and join dc. Then $abcd$ is the rectangle required.

For, by construction, $AB : AD :: ab : ad$; and since opposite sides in each rectangle are equal, the sides about each of the other angles are *proportional.*

176. Prop. LXXVII. *Upon a given straight line as a base to construct a rectilineal figure similar to a given rectilineal figure.*

(1) Since, by (89), two *similar* rectilineal figures of

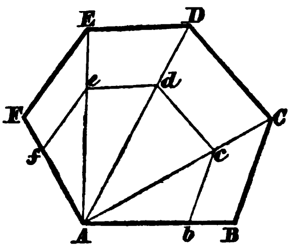


any number of sides may be divided into the same number of similar triangles, each to each, and similarly situated, let the given rectilineal figure, as $ABCDEF$, be divided into its component triangles by drawing the diagonals AC, AD, AE; and let ab be the given base on which it is required to construct a figure similar to $ABCDEF$. Upon ab construct the *triangle abc* similar to ABC (174). Then upon ac construct the triangle acd similar to ACD. Again upon ad the triangle ade similar to

ADE; and upon *ae* the triangle *aef* similar to *AEF*. Then the whole figure *abcdef* shall be *similar* to *ABCDEF*; and it is constructed upon the base *ab*.

For, since each of the triangles in *abcdef* is similar, and ∴ *equiangular* to the corresponding triangle in *ABCDEF*, it is easily seen that the *whole* figures *ABCDEF*, and *abcdef* are themselves *equiangular*. Also, the sides about equal angles are *proportional*, because these sides are the sides of similar *triangles*, and ∴ *as such* are *proportional* (71).

(2) Or, if the straight line, which is to be the base, be given in *magnitude* only, and not in position, draw

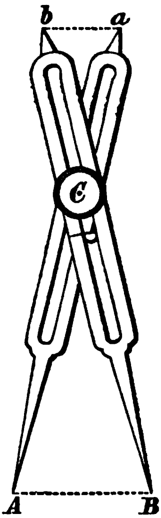


the diagonals *AC*, *AD*, *AE* as before; in *AB* take *Ab* equal to the given base; through *b* draw *bc* parallel to *BC* meeting *AC* in *c*; draw *cd* parallel to *CD* meeting *AD* in *d*; *de* parallel to *DE* meeting *AE* in *e*; and *ef* parallel to *EF* meeting *AF* in *f*. Then *Abcdef* is the figure required.

The proof is obvious from (71 Cor. 2).

177. Prop. LXXVIII. *To explain the construction and use of the* Proportional Compasses.

To facilitate the construction of *similar* figures a very useful instrument has been invented, called '*Proportional Compasses*'. It consists of two parts exactly equal, which are worked to a fine point *at both ends*, and are so fastened together, by means of a screw, (as seen in the annexed diagram) that they become a sort of double compasses, *Aa* being one of these parts, and *Bb* the other. Both limbs of the instrument have an equal groove or slit, in which the screw *C* moves, and in any position of *C* within this groove it can be firmly tightened, so as



to make the legs CA, CB invariable, and also Ca, Cb; at the same time permitting motion *round itself* like the hinge of ordinary compasses.—C being the *centre* of this screw or hinge, ACa is a straight line, and so also is BCb. When the instrument is used, the two limbs are first brought into exact juxtaposition, so that the points A and B coincide, and also a and b. Then, according to the requirements of the problem in hand, the centre C is fixed, making CA and CB bear a certain *proportion* to Ca and Cb. (This is done by means of a graduated scale on the instrument itself). Having thus fixed the legs in a certain proportion, *any* distance AB will bear *the same proportion* to ab, since ACB, aCb are always *similar* triangles. So that, opening the one pair of legs to embrace any given length or line, we have the required length on the reduced scale at once determined by the other pair; and the same may, of course, be done for *any number* of lines which are required to be in the same proportion.

There are several other uses to which this valuable instrument is put. For instance, it has a graduated scale upon it called '*circles*'; and this enables us so to fix the point C, that the circumference of the circle, whose radius is AB, shall be divided into any proposed number from 6 to 20 inclusive, of equal parts by stepping round it with the opening ab. We can do this, because there exists an invariable proportion between the side of a regular polygon of a given number of sides and the radius of the circumscribed circle (91).

And, *generally*, whenever an *invariable proportion* is known to exist between two particular geometrical magnitudes of a class, it is obvious that the Proportional Compasses may in such case be usefully employed.

Cor. If C be irremoveably fixed, so that CA and CB are each *double* of Ca and Cb; then in all cases ab will be *half* of AB, and we can *bisect* any given straight line, not too long, with great ease and accuracy. Not

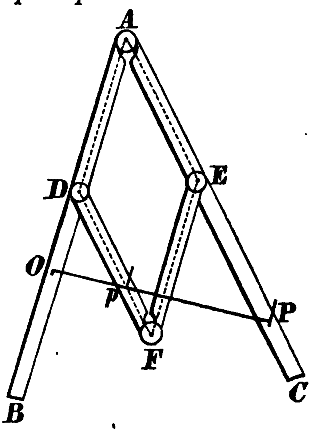
only so, but we can divide the line into any *even* number of equal parts, by successively taking the half of the last half, until the proposed number of parts is attained. There is such a simple form of the instrument, called '*Wholes and Halves*'.

178. PROP. LXXIX. *To explain the construction and use of the* PANTAGRAPH.

The PANTAGRAPH is an instrument used by draughtsmen for *copying* drawings, (that is, for constructing *similar* figures) upon the same scale, or a reduced scale, or an enlarged scale, as may be required; and when we say that, in good hands, it performs its work in each case with all attainable accuracy, *without the aid of either ruler or compasses*, the value of the instrument will at once be admitted. It has also this great merit, that it does not confine itself to *straight lines and circles*. Any curved line whatever presents no obstacle to its working So that the most crooked fence, once laid down on paper by the surveyor, can be copied exactly as it is, or on a smaller or larger scale, with ease and accuracy, at a single operation.

The best way to get a satisfactory knowledge of the instrument (and the same may be said of most other instruments) is to *see* it, and to see it *at work*. But it may be tolerably well understood from the following description of it, and of the *principle* of its construction.

AB, AC are two bars, or rulers, and DF, EF are two shorter ones, connected together, as shewn in the annexed diagram, by means of hinges at A, D, E, F, and so that the lines joining the centres of these hinges (the dotted lines) form a *parallelogram* $ADFE$. Thus the limbs of the instrument have free motion round the points A, D, F, E, but under no circumstances can $ADFE$ cease to be

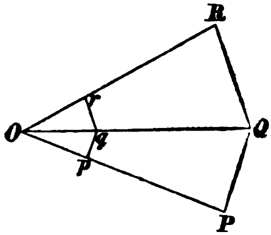


a *parallelogram.* When the instrument is used, it is laid *flat* upon the drawing board (like the ordinary parallel ruler), and it moves upon small castors placed beneath the points A, B, C, F. DB and DF are divided into the same number of parts, and the points of division are marked by figures to enable the draughtsman to set the sliding index, which is upon each of them, so as to produce the copy on the exact *scale* required. The sliding index is fixed by means of a clamped screw in each case, as at O and p; OpP is a straight line. Then, if the drawing is to be made on a *reduced* scale, a *tracer* is placed in a socket at P in the ruler AC, and a *pencil* in like manner at p; O is made the *fulcrum* round which the whole instrument moves, and *is the only fixed point in it.* The original drawing is then placed under the tracer P, and as this tracer is steadily made to traverse the outline of the drawing, the pencil p, which is in contact with the drawing-board, accurately traces out the required copy. If the copy is to be on a *greater* scale than the original drawing, it is only necessary for the tracer and pencil to exchange places. And if the copy is to be on the same scale, the pencil and the fulcrum exchange places.

That p *must* trace out a figure *similar* to that gone over by P will appear thus:—

$ADFE$ is always a parallelogram; $\therefore Dp$ is always *parallel* to AP; and $\therefore Op : OP :: OD : OA$. But O, D, and A are fixed points, $\therefore OD : OA$ is a fixed ratio; and $\therefore Op : OP$ is a fixed ratio, never varying throughout the operation.

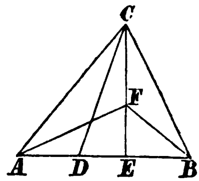
Suppose then the tracer, P, to move over a straight line PQ, during which time the pencil p traces out pq; then since $Oq : OQ :: Op : OP$, pq is *parallel* to PQ. Again let the tracer move over QR, while the pencil traces out qr; since $Or : OR :: Op : OP :: Oq : OQ$, $\therefore qr$ is *parallel* to QR. And so on to the end of the drawing; $\therefore pqr$, &c. is *similar* to PQR &c. (71); and it is on the proposed



scale, since each part of the perimeter *pqr* &c. : the corresponding part of *PQR* &c. :: *Op* : *OP*.

179. PROP. LXXX. *To construct a triangle whose area shall be to that of a given triangle in a given ratio.*

Let *ABC* be the given triangle; and $a : b$ the given ratio. Find *AD* a fourth proportional to b, a, and *AB*, so that *AD* is to *AB* in the given ratio (77). Join *CD*; and the triangle *ACD* shall be the triangle required.



For, since the areas of triangles between the same parallels, that is, of the same altitude, are proportional to their bases (73), the triangle *ACD* : triangle *ABC* :: *AD* : *AB* :: $a : b$.

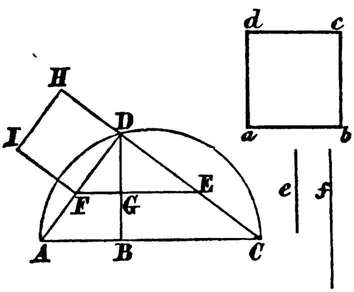
Or, if *CE* be drawn perpendicular to *AB*, and *EF* be taken a *fourth* proportional to b, a, and *CE*, so that *EF* : *EC* in the given ratio, and *AF*, *BF* be joined, the triangle *ABF* is the required triangle. For

EAF : *CAE* :: *EF* : *CE*, and *EBF* : *CBE* :: *EF* : *CE* (73),

∴ *ABF* : *ABC* :: *EF* : *EC* :: $a : b$ (80).

180. PROP. LXXXI. *To construct a square whose area shall be to that of a given square in a given ratio.*

Let *abcd* be the given square, and $e : f$ the given ratio. Upon any straight line take *AB* equal to e, and *BC* equal to f. Upon *AC*, as a diameter, describe a semicircle; from *B* draw *BD* at right angles to *AB*, meeting the circumference in *D*. Join *AD*, *CD*. In *DC* take $DE = ab$, and through *E* draw *EF* parallel to *AC*. Then upon *DF* construct the square *DFIH*, and *DFIH* is the square required.



For, if G be the point where *EF* meets *BD*, since ∠ *EDF* is a right angle, *DFG*, *DEG* are *similar* tri-

angles (72), and $DFG : DEG ::$ square of DF : square of DE (76). But $DFG : DEG :: FG : EG$ (73) $:: AB : BC$ (168); $\therefore$ square of DF : square of $DE :: AB : BC :: e : f$.

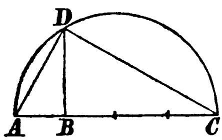
COR. 1. Since the areas of *circles* are proportional to the *squares* of their radii, or diameters (93), the same construction will serve to find a *circle* which shall be to a given circle in a given ratio; DE being taken equal to the radius, or the diameter, of the *given* circle, DF will be the radius or diameter of the *required* circle.

COR. 2. So also a polygon of any number of sides may be made having a given ratio to a given *similar* polygon, because *similar* polygons have their areas proportional to the *squares* of homologous sides (92). DE being taken equal to any side of the *given* polygon, DF will be the corresponding side of the *required* polygon, on which, as a base, the polygon may be constructed by (176).

N. B. When the copy of a drawing is said to be on a *reduced*, or an *enlarged*, *scale*, the reference is always made to *lineal* dimensions, that is, corresponding *lines* in the original and in the copy are in the stated ratio. But in the above proposition a method is pointed out of reducing the *area* also of any rectilineal figure or circle *in any given ratio*. The learner should carefully observe this distinction.

181. PROP. LXXXII. *To construct a square which shall be any proposed multiple of a given square.*

This was done in (122); but the following method is also deserving of notice.



Let AB be equal to a side of the given square; produce it to C, making BC the same multiple of AB, that the required square is to be of the given square. Upon AC, as a diameter, describe a semicircle; and from B draw BD at right angles to AB, meeting the circumference in D. The square of BD is the square required.

For, joining AD, CD, $\angle ADC$ is a right angle (54), and $\therefore ABD$, CBD are *similar* triangles (72);

$\therefore ABD : CBD ::$ square of AB : square of BD (76).

But $ABD : CBD :: AB : BC$ (73),

$\therefore$ square of AB : square of $BD :: AB : BC$;

and BC was *made* the proposed multiple of AB, $\therefore$ the square of BD is the required multiple of the square of AB.

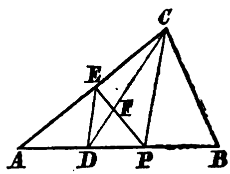
The fig. is here drawn so that the square of BD is equal to *three times* the square of AB.

COR. 1. Since the areas of *circles* are proportional to the *squares* of their radii, or diameters, this construction will serve to find the *circle* which is any proposed multiple of a given circle. AB being taken equal to the radius, or diameter, of the *given* circle, BD will be the radius, or diameter, of the *required* circle.

COR. 2. The same may be said of any two *similar* rectilineal figures; AB being equal to a side of one, BD is equal to the corresponding side of another, whose area is the same multiple of the former that BC is of AB.

182. PROP. LXXXIII. *To divide a given triangle into two parts, which shall be in a given ratio to each other, by a straight line drawn from a given point in one of the sides.*

Let ABC be the given triangle; P the given point in the side AB. Join PC; and divide AB in the point D, so that the parts AD, DB are in the given ratio (170). Join CD; draw DE parallel to PC, and join PE. The triangle ABC is divided by the line PE into the parts required.



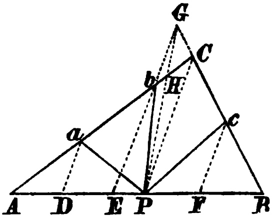
For, the triangle ACD : triangle $BCD :: AD : DB$ (73). Also, the triangles EDC, EDP are equal to one another, being upon the same base ED, and between the same parallels ED, PC, $\therefore$ the triangle $APE =$ the triangle ACD, and the quadrilateral $PBCE =$ the triangle BCD,

$$\therefore APE : PBCE :: AD : DB.$$

COR. If it be required to *bisect* the given triangle by a line through P, *bisect* AB in D, and proceed as before.

183. PROP. LXXXIV. *To divide a given triangle into any proposed number of parts, which shall be to each other in given ratios, by straight lines drawn from a given point in one of the sides.*

Let ABC be the given triangle; P the given point in the side AB. Join PC; and divide AB into the given number of parts, and having to one another the given ratios (170). Suppose the number of parts to be four, (for the process is the same, whatever the number be), and let AD, DE, EF, FB be the parts. Draw Da, Eb, Fc all parallel to PC; and join Pa, Pb, Pc. The triangle ABC is divided into the required parts by the lines Pa, Pb, Pc.



For, $AaP : abP :: Aa : ab$ (73),

$:: AD : DE$ (70).

Also, producing Eb, BC to meet in G, join PG intersecting AC in H. Then, since $PGC = PbC$ (41), the part $GHC =$ the part PbH; $\therefore$ the triangle $PGC =$ the quadrilateral $PbCc$.

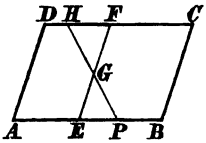
But $PGc : PBc :: Gc : Bc :: EF : FB$,

$\therefore PbCc : PBc :: EF : FB$.

COR. If it be required to divide the given triangle into a certain number of *equal* parts, divide AB into that number of *equal* parts, and proceed as before.

184. PROP. LXXXV. *To divide a given parallelogram into two parts, which shall be in a given ratio to each other, by a straight line drawn from a given point in one of the sides.*

Let $ABCD$ be the given parallelogram; P the given point in the side AB. Divide AB into two parts in the point E, such that $AE : EB$ in the given ratio (170). Through E draw EF parallel to AD or BC, and $\therefore$ dividing $ABCD$ into two *parallelograms*. Bisect EF in G. Join PG, and produce it to



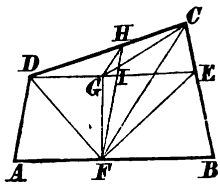
meet the side DC in H. PH is the straight line required.

For, PGE, FGH are equal triangles (39); $\therefore$ $PADH$ = the parallelogram $AEFD$; and $PBCH$ = the parallelogram $BEFC$. But $AEFD : BEFC :: AE : EB$ (73), $\therefore$ $PADH : PBCH :: AE : EB$.

Cor. If it be required to *bisect* $ABCD$ by a straight line through P, *bisect* AB in E, and proceed as before.

185. Prop. LXXXVI. *To divide any given quadrilateral figure into two parts, which shall be in a given ratio to each other, by a straight line intersecting two opposite sides.*

Let $ABCD$ be the given area. Draw DE parallel to AB, meeting BC in E. Divide AB and DE so that $AF : FB$ in the given ratio, and also $DG : GE$. Join FG, CG, FC. Draw GH parallel to FC, meeting DC in H; and join FH. FH is the straight line required.



For, if FH, GC intersect in I, and FD, FE be joined,

$$CDG : CEG :: DG : GE \text{ (73)},$$

$$\text{also } FDG : FGE :: DG : GE,$$

$$\text{and } FAD : FBE :: AF : FB :: DG : GE;$$

$$\therefore ADCGF : BCGF :: DG : GE \text{ (80)}.$$

But the part HCG = the part HFG (41);

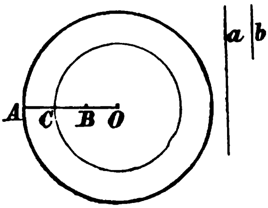
$$\therefore ADHF = ADCGF, \text{ and } BCHF = BCGF,$$

$$\therefore ADHF : BCHF :: DG : GE :: AF : FB.$$

Cor. To *bisect* the area AF must be taken *equal* to FB, and DG *equal* to GE.

186. Prop. LXXXVII. *To cut out of the middle of a given circle a smaller circle which shall be to the former in a given ratio.*

Let O be the centre, and OA any radius, of the given circle; $a : b$ the given ratio. Find OB a fourth proportional to a, b, and OA (77). Then find OC a *mean proportional* between OA, and OB (72 Cor. 2). With centre O, and radius OC, describe a circle, and it shall be the circle required.

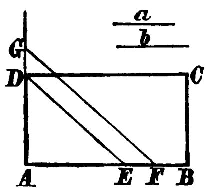


For, since the areas of circles are proportional to the squares of their radii (93), the greater circle : the smaller :: the square of OA : the square of OC. But the square of OC = the rectangle OA, OB (74); ∴ the greater circle : the smaller :: the square of OA : the rectangle OA, OB. Now the square of OA, and the rectangle OA, OB, being upon the same base OA, will be proportional to their altitudes, and will ∴ be in the ratio $OA : OB$;

∴ the greater circle : the smaller :: $OA : OB :: a : b$.

187. Prop. LXXXVIII. *To construct a rectangle which shall be equal to a given rectangle, and have its sides in a given ratio to one another.*

Let $ABCD$ be the given rectangle; and $a : b$ the given ratio. In AB take AE a *fourth proportional* to a, b, and AD (77); and AF a *mean* proportional between AB and AE, (72 Cor. 2). Join DE, and draw FG parallel to DE, meeting AD produced in G. Then the rectangle contained by AF, AG is the rectangle required.



$$\text{For } AB : AF :: AF : AE \text{ (72)},$$

$$\text{and } AF : AE :: AG : AD \text{ (70)};$$

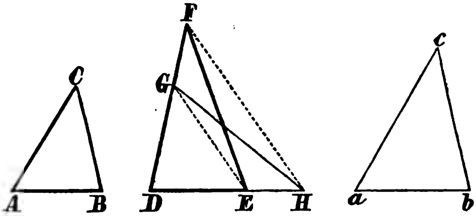
$$\therefore AB : AF :: AG : AD,$$

and ∴ the rectangle AB, AD = the rectangle AF, AG (74).

Also $AG : AF :: AD : AE :: a : b$.

188. Prop. LXXXIX. *To construct a triangle which shall be* similar *to one, and* equal *to another, given triangle.*

Let ABC, DEF be the two given triangles; and let



it be required to make a third triangle *similar* to ABC, and *equal* to DEF.

Place the two given triangles so that their bases AB, DE are in the same straight line. Through C draw CG parallel to AD, meeting DF, or DF produced, in G. Join EG; draw FH parallel to EG meeting DE, or DE produced, in H; and join GH. Then in AB produced take ab a *mean proportional* between AB and DH (72 Cor. 2); through a draw ac parallel to AC, and through b draw bc parallel to BC; abc is the triangle required.

For, since AB, and ab are in the same straight line, and ac, bc are respectively parallel to AC, BC, it is obvious that ABC, and abc are equiangular, and therefore *similar*, triangles. Also, to shew that abc is *equal* to DEF. Since EGF, EGH are *equal* triangles, being upon the same base and between the same parallels (41), $\therefore$ the triangle $DHG = DEF$. And ABC, DHG have the same *altitude*, since CG is parallel to AE;

$$\therefore \Delta^* ABC : \Delta DHG :: AB : DH \text{ (73)}.$$

And $\Delta ABC : \Delta abc$ :: square of AB : square of ab (76).

But the square of ab = the rectangle AB, DH (74),

$\therefore \Delta ABC : \Delta abc$ :: square of AB : rectangle AB, DH.

Now the square of AB and the rectangle AB, DH, being parallelograms of the same altitude AB, are proportional to their *bases*, that is,

square of AB : rectangle AB, DH :: $AB : DH$ (73);

$$\therefore \Delta ABC : \Delta abc :: AB : DH,$$

$$\text{and } \therefore \Delta ABC : \Delta abc :: \Delta ABC : \Delta DHG,$$

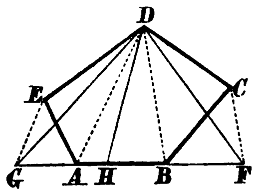
* This abbreviation, Δ for '*triangle*,' is allowable, if the student be careful not to confound it with $\angle$, which stands for '*angle*.'

which means that ΔABC *contains*, or *is contained* in, Δabc, the same number of times that it *contains*, or *is contained* in, ΔDHG (65). Hence it follows, that

$$\Delta abc = \Delta DHG = \Delta DEF.$$

189. Prop. XC. *To construct a triangle which shall be either equal to a given polygon, or be in any proposed* ratio *to it.*

(1) Let $ABCDE$ be the given polygon; (the number of sides is immaterial to the process); produce AB indefinitely both ways; join BD; through C draw CF parallel to BD, meeting AB, or AB produced, in F, and join DF. Join AD; and through E draw EG parallel to AD, meeting AB, or BA produced in G; and join DG. The triangle DGF shall be *equal* to the polygon $ABCDE$.



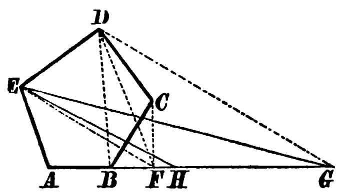
Or, if the required triangle is to bear a certain *ratio* to the polygon $ABCDE$, divide GF in H, so that $GH : FG$ is equal to that ratio (170), and join DH. Then DHG is the triangle required.

For BDC, BDF, being triangles upon the same base and between the same parallels, are equal to one another (41). Also $\Delta ADE = \Delta ADG$; $\therefore$ adding to these equals the ΔABD, it is evident that $\Delta DGF =$ the polygon $ABCDE$.

$$\text{Also } \Delta DHG : \Delta DGF :: GH : FG \text{ (73)},$$
$$\therefore \Delta DHG : ABCDE :: GH : FG.$$

(2) The same thing may be done, so as to *retain one side and one angle* of the polygon, thus;—

Proceed as before at first, but instead of joining AD, join EF, and through D draw DG parallel to EF, meeting AB produced in G; and join EG. EAG is a triangle *equal* to the polygon $ABCDE$. And if AG be divided in H, so that



$AH : AG$ in a proposed *ratio*, and EH be joined, then $\Delta AEH : ABCDE$ in that *ratio*.

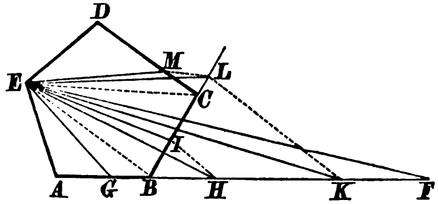
For, as before, $\Delta BDF = \Delta BDC$, $\therefore AFDE = ABCDE$. Again, $\Delta EFD = \Delta EFG$, $\therefore$ adding to these equals ΔAFE, it is obvious that $\Delta AEG = AFDE = ABCDE$. Also $\Delta AEH : \Delta AEG$, that is $ABCDE :: AH : AG$ (73).

Obs. Since a triangle is the simplest of all rectilineal figures enclosing a space, it is clear that to be able to reduce any rectilineal figure whatever to an *equivalent* triangle, that is, of *equal area*, must be of considerable practical importance. Accordingly, the above construction is constantly used by *Surveyors* in converting irregular plots of land into triangles for the purpose of measurement, as will be explained more at length in Part III. It may be worth while however to observe here, that *in practice* it is not necessary actually to *draw* any of the dotted lines—but simply to *mark* the points F, G, &c. on the *base* AB produced. For example, the *parallel-ruler* being placed with its outer edge to coincide with B and D, there is no need *to draw the line* BD, or even CF parallel to it, but simply to *mark the point* F, where that parallel meets the *base*. And so on, until we arrive at the final triangle.

Observe also, that this construction points out a correct method of cutting off corners of land, by introducing *one* straight fence instead of *two or more*, while the area remains the same. Thus DF will replace DC, and CB, without altering the area of $ABCDE$.

190. Prop. XCI. *To divide a given polygon of any number of sides into two parts, which shall be to one another in a given ratio, by a straight line drawn from the vertex of one of the angles.*

Let $ABCDE$ be the given polygon; and E the given



vertex. Produce AB indefinitely; and by the 2nd method of (189) draw the triangle EAF equal to the polygon $ABCDE$. Divide AF into two parts in the point G, so that $AG : GF$ in the given ratio, and join EG. Then if G fall in AB, EG divides the polygon as required.

$$\text{For, } \Delta EAG : \Delta EGF :: AG : GF \text{ (73);}$$
$$\text{and } \Delta EGF = EGBCD, \text{ since } EAF = ABCDE;$$
$$\therefore EAG : EGBCD :: AG : GF.$$

But, if in dividing AF in the given ratio, the point of division falls *beyond* B, as at H, join EH, EB, and through H draw HI parallel to EB, meeting BC in I, and join EI. Then, if I fall between B and C, EI divides the polygon as required. For $\Delta EBI = \Delta EBH$ (41), $\therefore EABI = \Delta EAH$; also $\Delta EHF = EICD$, $\therefore EABI : EICD :: AH : HF$.

Or, if in dividing AF in the given ratio, the point of division, as K, fall so far beyond B, that the parallel to EB, as KL, does not meet BC, but BC *produced*, as at L, then join EC, and through L draw LM parallel to EC, meeting CD in M. In that case EM divides the polygon as required. For $\Delta EBK = \Delta EBL$, $\therefore EABL = \Delta EAK$; also $\Delta ECM = \Delta ECL$, $\therefore EABCM = EABCL$.

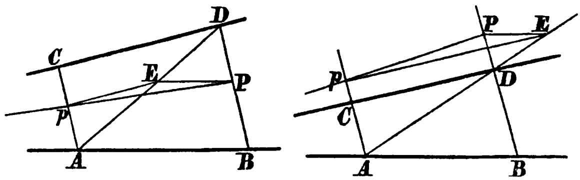
$$\therefore EABCM : EMD :: \Delta EAK : \Delta EKF :: AK : KF.$$

Cor. By following the same method, any given polygon may be divided into *any number* of parts, either equal to one another, or in given ratios. Thus, the polygon above is divided by EG, EI, and EM into four parts, which have to each other the same ratios that AG, GH, HK, KF have. And if AG, GH, HK, KF are taken *equal* to each other, then the polygon $ABCDE$ is divided into four *equal* parts.

191. Prop. XCII. *From a given point to draw a straight line, which would, if produced, pass through the point to which two given straight lines in the same plane converge, when the latter point is either inaccessible, or cannot be marked down on the Drawing-Board.*

Let AB, CD be the given straight lines; and P the given point from which the required line is to be drawn. Through P draw a straight line making, as nearly as

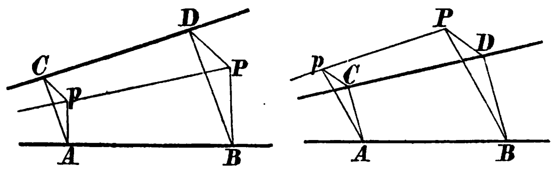
you can guess, equal angles with the given lines, and meeting them in B, and D; and draw another line AC parallel to BD, and as near to the unknown point of convergence as is convenient.



Join AD; through P draw PE parallel to AB, meeting AD, or AD produced in E. Through E draw Ep parallel to CD, meeting AC, or AC produced, in p. Join Pp, and Pp shall be the line required.

For, $DP : PB :: DE : EA :: Cp : pA$ (70) $\therefore$ since the parallel lines BD, AC are divided *proportionally* in P, p, the line Pp will coincide with the line drawn to P from the vertex of the opposite angle of the triangle whose base is BD and sides BA, DC produced (168).

Another Method. Instead of drawing BD through



P, draw it at some short distance from P, and join PB, PD. Draw AC parallel to BD, Cp parallel to DP, and Ap parallel to BP, the two last lines drawn intersecting each other in p. Join Pp, and it is the line required.

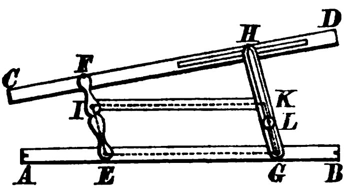
For, if BA, DC produced intersect at a point I, then $AC : BD :: AI : BI$ (71). Also ACp, BDP are similar triangles, as may easily be shewn; $\therefore$ $Ap : BP :: AC : BD$, and $\therefore$ $Ap : BP :: AI : BI$, from which it follows, that BpI is a straight line.

Either of the above constructions is sufficiently simple; but in certain cases, in perspective drawings for instance, a *great number* of lines converging to a

point not within the drawing is required, and it would be troublesome to repeat the construction over and over again. Consequently an instrument has been invented, something after the fashion of a *Parallel-Ruler,* to save all this trouble to the draughtsman. It is called the '*Centrolinead*'.

192. Prop. XCIII. *To explain the construction and use of the* Centrolinead.

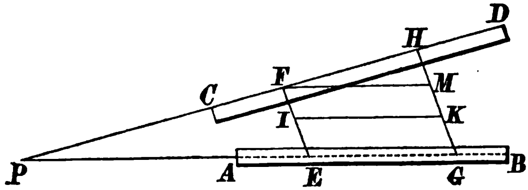
The *Centrolinead* is a ruler by which we are enabled to draw through any given point a straight line which, if produced, shall certainly pass through the point of intersection of any two or more *given* straight lines, when the latter point is not determined.



It consists of two flat-rulers *AB,* and *CD,* connected together by two bars *EF*, *GH*, moveable round joints at *E*, *F*, *G*, and *H*. *EF* is of a fixed length, which never varies in the same instrument, as in the *Parallel-ruler;* but *GH* is made up of two parts, one sliding on the other by means of a groove, so that *GH* admits of different lengths, and is adjusted to some particular length by a screw at *L,* which clamps the two parts of *GH* together, when required. *G* is a *fixed* point, but the point at *H* is carried in a groove, and slides on the ruler *CD*. There is also another ruler *IK,* fixed as in the diagram, with joints at *I* and *K, so* that *EGKI* is always a *parallelogram,* as shewn by the *dotted* lines. The *centres* of motion at *E* and *G* are *equidistant* from both edges of the ruler *AB*, and the points where *EG* produced meets the *ends* of the ruler are marked on the ruler. But the centres at *F* and *H* are so made, that they are *in a line* with the *edge* of the ruler *CD*.

In using the instrument, *AB* is first made to coincide with one of the two *given* converging lines (not the *edge* of *AB,* remember, but the two marks on the *ends* of *AB*, mentioned before). The screw *L* is slackened, and holding *AB* tight, *CD* is brought to coincide with the *second* given line. Then the screw is clamped tight; and while

AB is still held in its original position, the edge of *CD* can be made to pass through any proposed point within

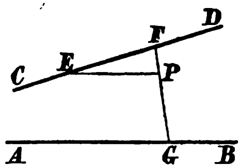


certain limits, and *all* lines drawn along it will converge to the same point as the two given lines.

For, suppose the instrument adjusted for a particular case; and *suppose* *BA*, *DC* produced to meet in *P*. From *F* draw *FM* parallel to *IK*, meeting *GH* in *M*. Then since *EGKI* is always a *parallelogram*, $\angle PEF$ is always equal to $\angle FMH$, and $\angle EPF$ to $\angle HFM$; $\therefore$ *PEF*, *HFM* are always *similar* triangles; and $\therefore PE : EF :: FM : HM$. But *EF* is a *fixed* magnitude; $FM = IK$, and is $\therefore$ fixed also; HM = the difference between *GH* and *EF*, and is $\therefore$ fixed, as long as the screw *L* is clamped. Hence, three out of four of the terms of the above proportion being *fixed* magnitudes, it is obvious that the remaining one, *PE*, must be fixed also; and $\therefore$ the point *P* is fixed; that is, in every position of the ruler *CD*, for the same adjustment, the line *DC* produced will pass through the one point *P*.

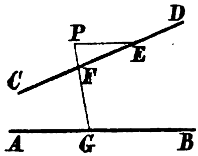
193. **Prop. XCIV.** *Having given two straight lines converging to a point, to draw through any given point in the same plane a straight line which shall make equal angles, on the same side of it, with the two given straight lines.*

(1) Let *AB*, *CD* be the two converging lines; *P* the given point through which the required line is to be drawn. Through *P* draw *PE* parallel to *AB*, meeting *CD* in *E*. On *CD* set off *EF* equal to *EP*; join *FP*, and produce it to meet *AB* in *G*. *FPG* or *PFG* is the line required.

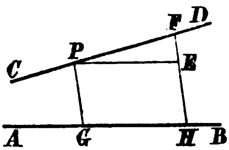


For, in the isosceles triangle PEF, $\angle EPF = \angle EFP$ (26); but $\angle EPF = \angle AGF$, since PE is parallel to AG (34 Cor. 4).

$$\therefore \angle AGF = \angle CFG.$$



(2) If the given point P be *in* one of the two converging lines, as CD, through P draw a line parallel to AB, and in it take *any* convenient length PE, and on CD set off PF equal to PE. Through E and F draw an indefinite straight line; and through P draw PG parallel to that line, meeting AB in G. PG is the line required.



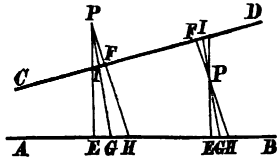
For $\angle PEF = \angle PFE$ (26); and $\angle AGP = \angle GPE = \angle PEF$ (34); also $\angle CPG = \angle PFE$ (34),

$$\therefore \angle AGP = \angle CPG.$$

Cor. The straight line PG or any straight line parallel to it within the range of the drawing, determines also two points, one in each of the given lines, *equidistant* from the undetermined point of intersection of the two given lines.

N. B. When the two given lines AB, CD converge to a point *very distant*, that is, are nearly parallel, the above construction, although theoretically correct, will be attended *practically* with much risk; and that for two reasons:—1st. because the point E will not be well determined by the intersection of two lines, which make a very small angle with each other; and 2nd. because in this case FP will be very small, and no very short line can be *produced*, or extended, in the ordinary way without chance of error. Hence in such a case some other method should be adopted, as the following:—

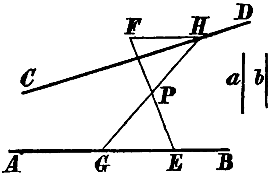
From the given point P draw PE, PF perpendicular to AB, CD respectively, and *at the same time* produce each of these perpendiculars indefinitely, if necessary; and let PF, or FP produced, meet AB in H. *Bisect* $\angle EPH$ by the straight line PG, meet-

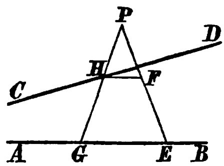


ing AB in G, and CD in I. Then GP, or GP produced, is the line required.

For, by construction, PFI, PEG are similar triangles, $\therefore \angle PIF = \angle EGP$, that is, $\angle AGI = \angle CIG$.

194. Prop. XCV. *Having given the same as in the last* Prop. *to draw through the given point* P *a straight line meeting the two given lines* AB, CD, *in* G *and* H, *so that* PG : PH *is a given ratio,* a : b.

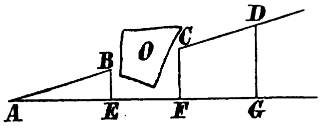




From P draw any straight line PE to AB; and, if P lie between AB, and CD, produce EP, and in EP produced, (or in PE, if P is *without* both AB and CD) take PF a *fourth proportional* to a, b, and PE. Through F draw FH parallel to AB, meeting CD in H. Join PH, and produce it to meet AB in G. GPH, or GHP, is the line required.

For PHF, PEG are similar triangles, $\therefore PG : PH :: PE : PF$ (71); and $PE : PF :: a : b$, by construction; $\therefore PG : PH :: a : b$.

195. Prop. XCVI. *To 'produce' a given straight line beyond an obstacle which prevents the application of the ruler, tape, chain, or other usual means by which straight lines are drawn.*



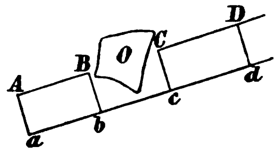
(1) Let AB be the given straight line, terminated at B on account of an obstacle O, but required to be continued, as CD, beyond the obstacle. Draw from A any indefinite straight line AG, which will clear the obstacle, and from B draw BE perpendicular to AG. Then in AG take any two other points F and G beyond the obstacle, and draw FC, GD at right angles to AG, making FC a *fourth proportional* to AE, BE, AF, and GD a *fourth proportional* to AE,

BE, AG. Join CD, and it shall be in the same straight line with AB.

For, if BC be *supposed* to be joined by AB produced, since BE is parallel to CF, $AE : BE :: AF : CF$, the same proportion as in our construction, $\therefore$ the point C is the same in both, that is, C is in AB produced. The same may be proved of D; $\therefore$ both C and D being in AB produced, the straight line CD is in it.

Obs. If $AEFG$ can be conveniently drawn making *half a right angle* with AB, then the points C and D will be determined by simply making $FC = AF$, and $GD = AG$. For, in that case, each of the triangles is *isosceles*.

(2) Or, if it be convenient, the following simple method may be adopted. From A and B draw Aa, Bb at right angles to AB, and equal to one another, and of such length that ab produced will clear the obstacle. In ab produced take two other points c, d, beyond the obstacle, and draw cC, dD at right angles, making $Cc = Dd = Aa = Bb$; and join CD. Then CD is the line required.

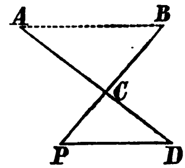


For, since the points B and C are equidistant from bc, $\therefore$ the straight line BC, if drawn, would be parallel to bc (35). Similarly AB is parallel to ab, and CD to cd. But ab, bc, cd are in the same straight line, $\therefore$ also AB, BC, CD are in one straight line.

(3) If the obstacle O be such, that two points in AB, as A and B, are *visible* from C and D, the proper position of C and D will be easily determined *by the eye*, as explained in (98). In which case CD being joined will be the line required; that is, it will be in the same straight line with AB.

196. Prop. XCVII. *Through a given point to draw a straight line parallel to another straight line, which latter line cannot be traced, but has two points in it only given.*

Let A and B be the two given points in the untraced line; P the given point through which a straight line is required to be drawn parallel to the straight line joining A and B.



Join PB, B being the more distant from P of the two given points, and divide BP in any convenient ratio in the point C. Join AC, and produce it to D, so that $AC : CD :: BC : CP$, and join PD. Then PD is the straight line required.

For, the triangles ACB, PCB have $\angle ACB = \angle PCD$ (31), and the sides about these equal angles *proportionals*, $\therefore$ the triangles are *similar* (71 Cor. 1), and $\therefore$ equiangular; $\therefore \angle ABC = \angle BPD$, and PD is *parallel* to AB (34).

Or, if it be more convenient to *bisect* PB in C, CD must be taken *equal* to AC, and PD will be parallel to AB, as before.

ARCHITECTURAL MOULDINGS, ARCHES, &c.

197. The profile of all Architectural mouldings is made up of straight lines and arcs of circles*; and this profile traced upon paper is called the 'working drawing', because by it the workman executes the design of the Architect. It is important therefore that these '*working drawings*' be constructed *upon right principles*, whatever those principles may be; and although it is no business of ours, as *Geometricians*, to discuss the *fundamental principles* of *Architecture*, yet there are two *rules* of *construction* so often observed, (with however many exceptions) that it seems desirable to point them out in the following examples both of *Mouldings* and *Arches*. These rules are,

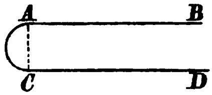
I. When an arc of a circle runs into a straight line, at a certain point, so that the straight line is, as it were, a continuation of the arc, that straight line should be a *tangent* to the circle at that point. And

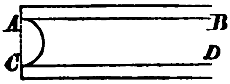
* For no other reason probably than the comparative ease with which straight lines and circles are put to use in constructive art.

II. When an arc of one circle runs into an arc of another, the two arcs should have *the same tangent* at the point of junction.

It follows as a consequence of Rule II, that *the centres of the circles and the point of junction will be in the same straight line* (143).

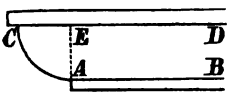
As a simple instance of the former Rule either of the following mouldings is of common occurrence:—

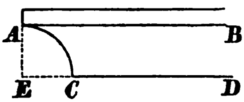




in both which $\angle BAC = \angle ACD =$ a right angle, and upon AC as diameter a *semi-circle* is described; $\therefore$ AB is a *tangent* to the circle at A, and CD at C.

198. There are other simple mouldings for which a *quadrant* is described instead of a semicircle, such as the following:

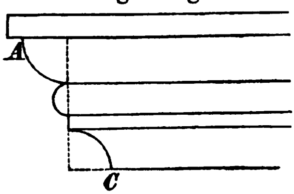




The construction is obvious. E is the *centre* of the circle of which AEC is a quadrant. AB *touches* the quadrant at A; and the arc AC *commences*, or *terminates*, at C with its tangent *at right angles* to CD, thus distinctly repudiating, as it were, a *continuation* of the curve in that direction.

Observe how the rule in question is carried out in the annexed compound moulding:—

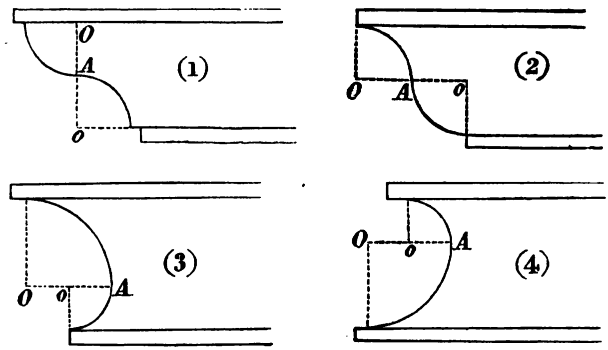
The moulding *begins* and *ends at right angles* to the horizontal line, at A and C. It consists of several distinct parts—but observe how each of these parts is *tangential* with those next to it. May it not be, that this neighbourly union of the parts is that, which gives



fitness and repose, and therefore, beauty to the whole? Let Architects decide.

It is further to be observed that in this example the *centres* of the *three* circles are in one vertical straight line.

199. The next class of mouldings requires *two* quadrants, so joined together that they have a *common tangent* at the point of junction. The quadrants may be equal as in figs. (1) and (2), or unequal as in figs. (3) and (4). *A* is the point of junction of the arcs; and *OAo* is a straight

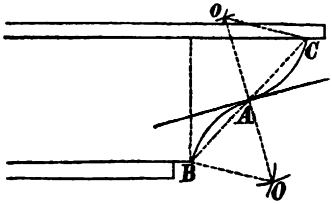


line. Hence the tangents to the two arcs at *A* coincide, being at right angles to the same straight line *OAo* (143).

The mode here shewn of joining the arcs of two different *quadrants* so as to produce one *continuous* curve is of frequent application in the Arts. The object is so to join them, that they may appear to belong to each other; and unless this be done by making them have the same *tangent* at the point of junction, the compound curve will appear *broken* at that point, and consequently offensive.

It is true, that Mouldings are frequently met with in which the circular arcs meet the *horizontal lines* neither as *tangents* nor *at right angles*, each arc being *less than* a quadrant; but even in such cases Rule II. is *always* observed in good examples, that is, the arcs meet *each other tangentially*.

The annexed diagram represents a case of this kind. BC is a straight line bisected in A. Upon AB, AC two equilateral triangles AOB, AoC are constructed.

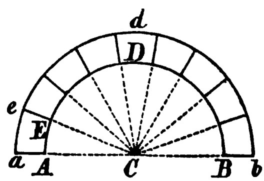


Then the *arc* AB is described from centre O, and *arc* AC from centre o. And because $\angle BAO =$ one-third of two right angles $= \angle CAo$, $\therefore OAo$ is a straight line (31); and it passes through the two centres and the point of junction A, $\therefore$ the arcs *touch* each other at A.

ARCHES are of various kinds, and serve various purposes, which it does not fall within our province to explain. But we will proceed to describe the '*working drawings*' from which the Arches in most common use are constructed.

200. PROP. XCVIII. *To draw a semi-circular arch of given span.*

Take the straight line AB to represent the given span. Bisect it in C; and with centre C, and radius AC, describe a semi-circle ADB. Increase this radius by Aa, or Bb, the depth of the arch-stones, and describe another semi-circle adb from the same centre. Divide the inner semi-circumference into an *odd** number of equal parts, according to the number of stones which are to form the arch; and join the several points of division with the centre C: produce these lines to the outer circumference, and they will determine the *joints* of all the arch-stones. Aa, Ee being two *contiguous* joints thus



* *Odd*, because on statical grounds it is not well to have any joint *vertical*, as would happen, if an *even* number were chosen.

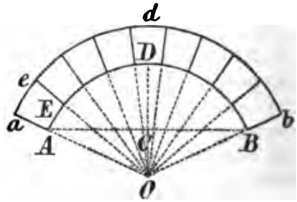
determined, *AEea* is the model for each one of the arch-stones, which are equal and similar in all respects.

Obs. In practice it is not *necessary* that the *outer* boundary, *adb*, be strictly *semi-circular*. Provided *Aa*, *AE*, and *Ee* are constructed as above, the actual boundary beyond *ae* may be of any form we choose.

If this arch be placed so that *AB* is horizontal, and the piers on which it rests are *vertical*, it obeys Rule I, (197), the straight lines which continue the arch downwards from *A* and *B* being *tangents* to the circle at those points.

201. Prop. XCIX. *To draw a segment-arch of given span and rise.*

Let *AB* be the given span; and *CD*, drawn at right angles to *AB* from its middle point, the given rise. Find *O* the *centre* of the circle which passes through the three given points *A*, *B*, *D*. With this centre, and radius *OA*, describe the arc *ADB*. Join *OA*, *OB*, and produce them to *a*, *b*, making $Aa = Bb =$ depth of arch-stones. With the same centre, and radius *Oa*, describe the arc *adb*. Divide *ADB* into an odd number of equal parts; and join the several points of division with the centre *O*. Produce each of these lines to meet the outer arc, as *OE* to *e*, and they will determine the *joints* of all the arch-stones. *Aa*, *Ee*, being two *contiguous* joints thus determined, *AEea* is the model for each one of the arch-stones, which are equal and similar in all respects.



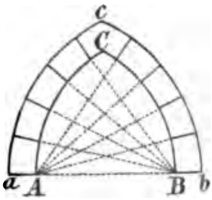
Obs. The first observation appended to Prop. XCVIII. applies here also; but not so the second. When this arch is connected with *vertical* piers or abutments, it will appear *broken* at *A* and *B*, because the vertical lines through those points will not be *tangents* to the circle. Its advantage over the *semi-circular* arch is obviously that of giving the same span with a much less rise, which is a great gain often in the case of bridges and other similar constructions having roads or canals over

them. The disadvantage is in the side-thrust outwards at A and B, tending to throw down the piers or abutments. But this is a point not to be explained here, as it belongs to the subject of *Mechanics*.

The arches in the last two propositions are said to be *struck* from *one centre*. There are, however, other kinds of arches struck from *two, three,* and sometimes *four,* centres. We will describe them in order.

202. PROP. C. *To draw a two-centred arch of given span.*

Let AB be the given span. With centres A and B, and radius AB, draw two arcs of circles intersecting in C. Produce AB both ways to a, and b, making $Aa = Bb =$ the depth of the arch-stones; and with the same centres A, and B and radius Ab, describe two other arcs bc, ac, intersecting in c. Then the arch AC is *centred* to B, and BC to A, that is, the *joints* of the arch-stones converge to those centres respectively. The top stone, or key-stone, as it is called, of the arch, is *generally*, though not always, made saddle-shaped, as shewn in the figure, to avoid the objectionable *vertical* joint at C.



This form of the *two-centred* arch is the one most commonly used, where the straight lines joining A, B, and C, form an *equilateral triangle*. But it is not *necessary* that the centres be at A and B. They may be in AB produced both ways; or the centre of BC may be any where in AB between A and the middle point; and the centre of AC between B and that middle point. The centres *must* be both in AB, or in that line produced, and *equidistant from* A and B, if not coincident with A and B. If it be desired to keep down the *crown* of the arch the centres will be taken between A and B. If, on the other hand, the height of the arch is to be increased, the centres will be taken in AB *produced*.

The effect of taking the centres in AB, or AB produced, obviously is to make the arch at its lowest points A, and B, obey Rule I. (197). But it is to be observed, that Rule II. is infringed at C and c, without producing

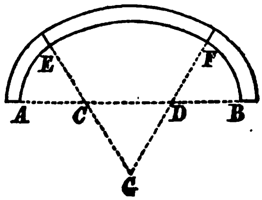
in this case any bad effect, on account of the arcs from those points each way downwards being perfectly *equal* and *symmetrical.*

This arch is more easily constructed than most others, but is seldom used (except in ecclesiastical structures, for which it is supposed to possess a peculiar fitness) on account of its great height in proportion to the span.

It is commonly called the *pointed* arch of two centres.

203. PROP. CI. *To draw a three-centred arch of given span.*

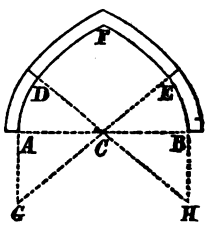
1. Let AB be the given span. From each extremity cut off equal portions AC, BD, each less than half AB. With centres A and C, and radius AC, describe intersecting arcs in E: and similarly with the same radius, and centres B and D, describe intersecting arcs in F; at the same time drawing the arcs AE, BF as portions of the required arch. Join EC, and FD, and produce them to meet in G. Then with centre G, and radius GE, or GF, draw the arc EF; and $AEFB$ shall be the inner boundary of the arch—that is, $AEFB$ shall be one continuous curve.



For, since the centres C and G are in the same straight line passing through the point of junction of the two arcs AE, EF, $\therefore$ AE and EF have the same tangent at the point E: similarly BF, and FE have the same tangent at F. Hence $AEFB$ is a continuous curve.—The joints of the stones from A to E are centred to C; from B to F they are centred to D; and from E to F they are centred to G.

2. Another form of a three-centred arch is constructed as follows:—

Bisect AB in C; with centre C and radius CA, draw *equal* arcs AD, BE. Join DC, EC; and produce them to meet AG, BH, which are at right angles to AB, in G and H. Then with centres G and



H, and radius GE, or HD, draw the arcs DF, EF meeting in F. $ADFEB$ is the inner boundary of the arch required. The stones from A to D, and from B to E, are centred to C. Those from D to F are centred to H; and those from E to F are centred to G.

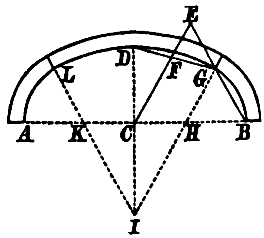
The proof is too obvious to need repetition.

This is a *pointed* arch of *three* centres.

204. PROP. CII. *To draw a three-centred arch of which both the span and rise are given.*

In the preceding Prop. the *span* only was given, and we were restricted to no particular *rise*. Here both span and rise are fixed for us.

Let AB be the given span, and CD, at right angles to AB from its middle point C, the given rise. Upon CB as a base describe the equilateral triangle BEC, on that side of BC on which the arch is to lie. In CE take $CF = CD$; join DF, and produce it to meet BE in G. Through G draw GHI parallel to EC, meeting BC in H, and DC produced in I. Make AK in AC equal to BH; join IK, and produce it indefinitely. Then with centres H and K, and radius BH, or AK, draw arcs BG, AL; and with centre I, and radius ID, draw the arc GDL.—$ALDGB$ shall be the inner boundary of the arch required.



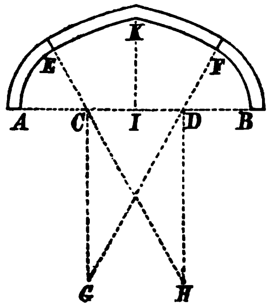
For, $IG = ID$, since $CF = CD$ (71); and the arcs AL, DL have the same tangent at L, since their centres K, I, and the point of junction L, are in the same straight line. Also arcs BG, DG have the same tangent at G, for the same reason; $\therefore$ $ALDGB$ is one continuous curve of *three* centres.

In constructing the arch the joints of the stones from A to L are centred to K; those from B to G are centred to H; and those from G to L are centred to I.

205. PROP. CIII. *To draw a four-centred pointed arch of given span.*

Let AB be the given span: from each extremity mark off equal parts AC, BD, each less than half AB, with centres C and D describe arcs AE, BF, making

them equal by describing intersecting arcs with the same radius from centres A and B. Join EC, FD; and produce them to meet DH, CG, which are at right angles to AB, in H and G. With centres G, H, describe the arcs FK, EK intersecting in K; and $AEKFB$ shall be the inner boundary of the arch required.

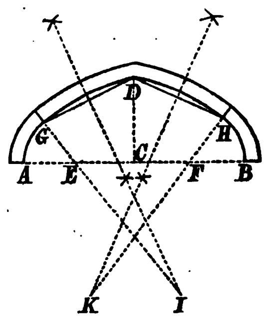


The proof is obvious from the preceding Propositions.

Obs. The above form of arch is the one most common of this class—but it is not absolutely necessary either that AEC, BFD should be *equilateral* triangles, or that the points G, H in FD and EC produced should be taken, and none other, for centres. *Any* equal arcs AE, BF, and *any* other points in FD and EC produced, *equidistant from C and D*, and beyond their point of intersection, will satisfy the Problem.

206. Prop. CIV. *To draw a four-centred pointed arch of which both the span and rise are given.*

Let AB be the given *span*, and CD the given *rise*. Mark off equal parts, in AB, from A and B, viz. AE, BF, and with centres E and F describe *equal* arcs AG, BH. Join GE, HF, and produce them indefinitely. Join DG, DH; bisect GD by a straight line cutting it at right angles (101), and produce this line to meet GE produced in I. Similarly, let the line which bisects DH at right angles meet HF produced in K.



With centres I, and K describe arcs GD, HD, meeting in D. Then $AGDHB$ is the arch required.

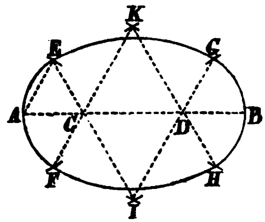
For, that it is a continuous curve is obvious from the preceding propositions. Also, since GD is bisected by a straight line passing through the centre, $\therefore$ GD is a *chord*

of the circle (49); that is, the arc described with centre I, and radius IG, passes through D. Similarly the arc described with centre K, and radius KH, passes through D.

207. PROP. CV. *To construct an* Oval*, *that is, a plane figure with curved symmetrical boundary returning into itself, not a circle, but composed of arcs of two or more circles forming one continuous curve.*

Various methods are usually given of describing an *Oval*, according as it is required, that the complete figure approach to, or recede from, a circle. But the following method includes them all:—

Take any straight line AB, and from each extremity mark off equal parts AC, BD, each less than half AB, and greater or less, according as the oval is to approach more or less to a *circle*. With centres A and C, and radius AC, describe two pairs of intersecting arcs, on opposite sides of AB, at E and F; and at the same time



draw the arc EAF. Similarly, with centres B and D, and radius BD, draw the equal arc GBH. With centres C and D, and radius CD, describe two pairs of intersecting arcs, on opposite sides of AB, at I and K. Then with centres I and K, and radius IE, draw the arcs EG, FH; and $AEGBHF$ is the *Oval* required.

For, joining AE, EC, IC, ID, since the triangles AEC, CDI are both equilateral, $\therefore \angle ACE = \angle ICD$, and $\therefore ICE$ is a straight line (31). Hence the arcs AE, EG have a common tangent at E. In the same manner, it will appear, that the arcs meeting at F, G and H, have also a common tangent at those points; that is, the curved line $AEGBHF$ is *continuous*, as required.

OBS. It is plain that the figure is divided into two equal and symmetrical parts by the line AB. Also, if a straight line be drawn through the middle point of AB,

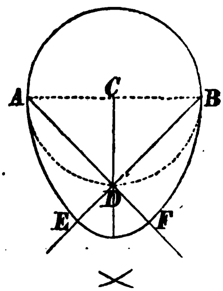
* From the Latin '*ovum*', an egg, the profile of an egg being *something* like what is meant by an *oval*.

at right angles to AB, and terminated by the curve, that line also will divide the figure into two equal and similar parts. These lines are called the *axes* of the figure—and if it be required to construct an Oval with *given axes*, that is, of given length and breadth (as we may say), it will be only necessary to repeat the process for constructing an arch of three centres, according to the first method in (203), on *both* sides of the greater *axis* taken as the span.

208. PROP. CVI. *To construct an egg-shaped oval.*

[This form of oval, which, strictly speaking is the only true oval, is of considerable practical value. Already it is allowed to be the best form for drain-pipes and sewers; possibly also many uses, to which it will hereafter be put, are not yet discovered.]

(1) On AB as a diameter describe the circle ABD, CD being the radius which is *at right angles* to AB. Join AD, BD, and produce them indefinitely beyond D. With centre A, and radius AB, describe the arc BF, terminated by AD produced in F. And with centre B, and the same radius, describe the arc AE, terminated by BD produced in E. With centre D, and radius DE, or DF, describe the arc EF. $ABFE$ shall be the oval required.



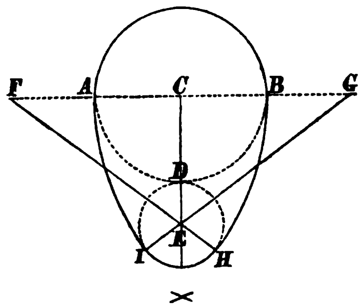
For, since the centres C and B of the arcs which meet at A are in the same straight line passing through that point, ∴ they have a common tangent at A. Similarly AE and FE have a common tangent at E; FE and FB at F; BF and BA at B.

(2) Or, if it be required to make the oval *longer* in proportion to the breadth, produce AB both ways, and take the centres for the arcs, which are to be drawn from A and B, somewhere in the parts of AB produced, equidistant from C, and join those centres with some point in CD produced.

The following particular construction will be easily remembered:—

$$AF = AC = CB = BG; \quad DE = \text{half of } AC.$$

Join FE, GE, and produce them. With centres G and F draw the arcs AI, BH terminated by GE, and FE, produced. With centre E, and radius EI, draw the arc IH, which completes the figure; and it is a *continuous* curve for the reasons stated in the first case.

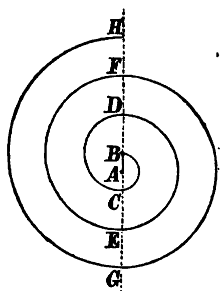


209. Prop. CVII. *To construct a* Spiral, *composed of arcs of circles of various radii.*

Def. A spiral is a curve described about a point from which it commences, and makes any number of circuits round that point without returning into itself.

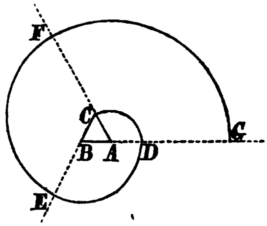
Of such curves there are an endless variety; but we are concerned only with those which are composed of *circular arcs*, of which we present the following four examples:

(1) Let AB be a given short straight line; produce it indefinitely both ways. With centre A and radius AB, describe a semi-circle; and let it meet BA produced in C. With centre B, and radius BC, continue this semi-circle by drawing another, which meets AB produced in D. With centre A, and radius AD, draw another semi-circle DE; and again with centre B, the semi-circle EF. And so on; alternately using for centres the given points A and B.



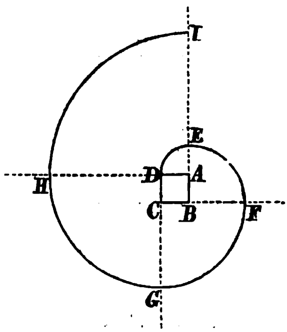
That the curve thus described is a *continuous* curve is obvious, because no two arcs are joined together in it except in the straight line which joins their centres.

(2) Another form of *Spiral* is constructed as follows:—*ABC* is a small equilateral triangle; produce *BA, AC, CB* indefinitely. With centre *A*, and radius *AC*, draw the arc *CD*, meeting *BA* produced in *D*. With centre *B*, and radius *BD*, draw the arc *DE*, meeting *CB* produced in *E*. With centre *C*, and radius *CE*, draw the arc *EF*, meeting *AC* produced in *F*. And so on; taking for centres the points *A, B, C in order*.



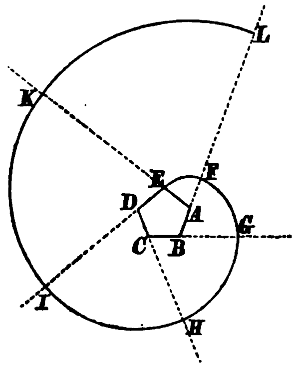
That the curve thus described is a *continuous* curve will appear for the reason stated in the first case.

(3) A third form of *Spiral* is constructed thus:—*ABCD* is a small *square*. Produce the sides *BA, CB, DC, AD* indefinitely. With centre *A*, and radius *AD*, draw the arc *DE*, meeting *BA* produced in *E*. With centre *B*, and radius *BE*, draw the arc *EF*, meeting *CB* produced in *F*. With centre *C*, and radius *CF*, draw the arc *FG*, meeting *DC* produced in *G*. With centre *D*, and radius *DG*, draw the arc *GH*, meeting *AD* produced in *H*. And so on; taking for centres the points *A, B, C, D in order*.



(4) Again, let *ABCDE* be a regular pentagon; and produce the sides *BA, CB, DC, ED, AE*, indefinitely. With centre *A*, and radius *AE*, draw the arc *EF*, meeting *BA* produced in *F*. With centre *B*, and radius *BF*, draw the arc *FG*, meeting *CB* produced in *G*. With centre *C*, and radius *CG*, draw the arc *GH*, meeting *DC* produced in *H*. With centre *D*, and radius *DH*,

draw the arc HI, meeting ED produced in I. With centre E, and radius EI, draw the arc IK, meeting AE produced in K. And so on, *ad libitum*, taking for centres the angular points A, B, C, D, E successively *in order*.



It is evident, that, by using any other regular polygon in the same manner for the initial figure, a different *spiral* will be formed; and so there is no limit to the number and variety of such curves.

Obs. It is worthy of observation that each of the above curves may be traced out, in practice, in a very simple manner without either compasses or ruler. Take the first case. At A and B, being points in the plane surface on which the spiral is to be traced, fix two pegs or pins. Round these, from one to the other, let a fine thread be wound, proceeding from left to right, one end of the thread being made fast, and the other terminating with a loop, in which a pencil or marker can be inserted, at B. Now unwind the thread, *taking care that it be always stretched tight*, and the marker in the loop will trace out the spiral.

For, at first the marker will trace a circular arc round A, until it arrives at C; then the centre will change to B, while a semi-circle is traced to D; then again the centre will be at A; and so on, precisely as the spiral was described by the compasses and ruler.

Similarly, if pegs be fixed at the angular points of the triangle, square, or pentagon, in the other cases, and a thread wound round them, the spiral in each case may obviously be traced by unwinding the thread, *provided it be kept tightly stretched throughout the operation.*

The '*Ionic Volute*' is a curve of similar character to the above, but too complex in its construction for an elementary work like the present.

TESSELATED PAVEMENT, AND INLAID WORK.

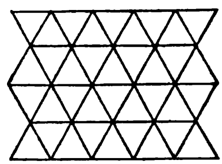
210. In order that any plane surface may be entirely covered by plane figures without either overlapping each other, or leaving gaps between them, it is necessary that the figures be such, and such only, that the sum of three or more of their angles is exactly equal to four right angles (30 Cor. 2).

That there must be *three* angles at least to fill up a space round a point is plain, because no angle can be so great as *two* right angles, and therefore no *two* angles whatever can amount to four right angles. Hence,

PROP. CVIII. *To find what regular rectilineal figures will exactly fit together, so as to cover any plane surface.*

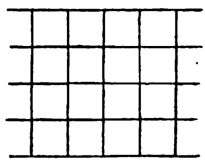
Take the regular figures in order according to the number of their angles:—

1st. *The Equiangular Triangle:*—In this case each angle is equal to *one-third* of two right angles, that is, *one-sixth* of *four* right angles. Therefore *six* such angles will exactly make up four right angles; and the equilateral triangle is such a figure as is required.



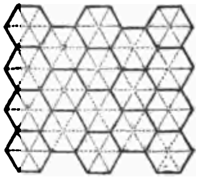
This combination of equilateral triangles is represented in the annexed diagram.

2nd. *The Square or Rectangle:*—In this case each angle is a right angle; so that *four* such angles make four right angles; and ∴ both the square, and rectangle, is such a figure as is required.



3rd. *The regular Pentagon:*—In this case each angle is equal to *six-fifths* of a right angle (86 Cor. 1); ∴ *three* such angles fall short of, and *four* exceed, four right angles; so that the regular pentagon is not such a figure as is required.

4th. *The regular Hexagon:*—In this case each angle is equal to *four-thirds* of a right angle (86 Cor. 1); ∴ three such angles are exactly equal to four right angles. Hence the regular hexagon is such a figure as is required.



This, in fact, follows from the first case, because each regular hexagon is made up of six *equilateral triangles* (160), as shewn by the dotted lines.

This combination of hexagons is remarkable as being the form adopted by bees in framing the honey-comb.

5th. *Regular Polygons of a greater number of sides:*—Since each angle of a regular polygon evidently *increases* as the number of sides increases; and since *three* angles of a regular *hexagon* are *equal* to four right angles; ∴ three angles of every *other* regular polygon with a greater number of sides must *exceed* four right angles. Hence no *other* regular figures exist, for the purposes here required, except those already determined, viz. the equilateral triangle, the square, and the regular hexagon.

Cor. It follows from the first case that a plane area may be covered by *lozenges*, whose shorter diagonal is equal to a side; as is often seen in the glazing of windows.

211. Prop. CIX. *To find what pairs of regular rectilineal figures, on the same base*, will exactly cover a plane surface.*

Since each angle of an equilateral $\Delta = \frac{2}{3}$ of a right angle,
.... a square $= 1$ right angle,
.... a hexagon $= \frac{4}{3}$ ....,
.... an octagon $= \frac{3}{2}$ ....;

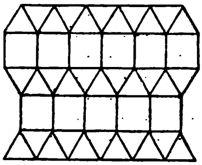
therefore,

(1) 3 angles of Δ + 2 angles of square $=$ 4 right angles;
(2) 4 angles of Δ + 1 angle of hexagon $=$ 4 right angles;
(3) 2 angles of Δ + 2 angles of hexagon $=$ 4 right angles;
(4) 2 angles of octagon + 1 angle of square $=$ 4 right angles.

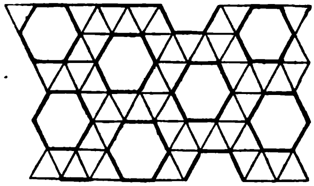
These combinations of two figures will be represented as follows:—

* That is, the *side* of the triangle = the side of each of the other figures.

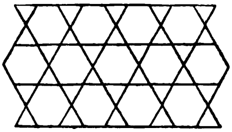
(1) The 1st thus:



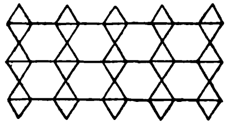
(2) The 2nd thus:



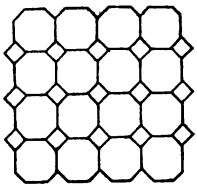
(3) the 3rd thus:



(3) or thus:

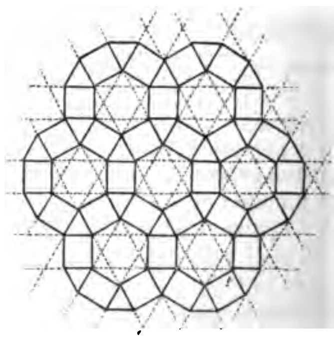


(4) the 4th thus:



212. Prop. CX. *To find what combination of* three *regular rectilineal figures, on the same base, will exactly cover a plane surface.*

There appears to be only one such combination, viz. 1 angle of $\Delta + 2$ angles of square $+ 1$ angle of hexagon, the sum of which is exactly 4 right angles. This combination will appear thus:



This beautiful arrangement of three regular figures, although complex in appearance, may be constructed as a pavement with remarkable ease and certainty, by first marking down the dotted lines, as in the diagram; and may be extended readily to cover any extent of plane area whatever.

213. Of course by deviating from *regular* figures other arrangements may be made differing from those here given; for instance, the second case in (210) may be changed from *squares* to *rectangles* in an endless variety. So also in the 1st case of (211). And again, in the 4th case of (211), the octagons *need* not be *equilateral* provided they are *equiangular*, and equal to one another.

A very pleasing combination is made by adding to the 4th arrangement in (211) a narrow strip, as a border, to each octagon, composed of *equal trapeziums*, rectangular at one end, with the other two angles equal to one right angle and a half, and half a right angle, respectively; the shorter base of the trapezium being equal to the height of the trapezium + a side of the square.

PRACTICAL HINTS AND DIRECTIONS FOR YOUNG DRAUGHTSMEN.

The Student will probably have gathered for himself, as he went along, many of the following notions; nevertheless it will not be without its use to recapitulate them here, in order that they may be the more strongly impressed upon his memory.

(1) See that you have good *Tools* to work with—that Rulers and Squares are *correct*—Compasses *sharp-pointed*—and Parallel Ruler well *tested*.

(2) Recollect it is not an easy matter to draw a straight line *perfectly straight*. Flat-rulers are mostly *feather-edged*, to enable you to draw lines *with ink, without blotting*, by raising the edge *above* the paper. This is a source of error, where great accuracy is required. It is always better to use a *pencil first*, so that the edge, along which it is drawn, may be *in contact* with the paper. Use *hard* pencils, and keep their points *fine*.

(3) In using the Compasses hold them *erect*, move them with a gentle hand, and pierce the drawing paper as

little as possible. A *large hole* cannot be the centre of a circle, or of an arc of a circle.

(4) Never *guess at* a straight line which is required to be at right angles, or parallel to, a given line. And, mark, the *shorter* the line is, the *less* likely are you to guess right.

(5) Never use the *end* of the flat-ruler to draw a straight line at right angles to a given line *from a given point in it*. Use the ruler with *cross-line* at right angles to both edges.

(6) In using the flat-ruler, or parallel-ruler, be careful to hold it tight to the drawing paper *with two fingers* at least; for if *one* finger only be pressed upon the ruler, it will probably, and perhaps imperceptibly, revolve round that point to a small extent, as the pencil or pen is passed along its edge.

(7) Keep the joint of your compasses at a medium stiffness, neither so tight as to require much force in moving the legs, nor so slack as to prevent them from being handled without altering the angle between the legs.

(8) Use the compasses, where practicable, in preference to the flat-ruler or straight-edge, for the sake of accuracy. A point is in no way more correctly determined than by intersecting arcs, on which account constructions made with the compasses alone are generally most accurate. The only exception is, when the intersecting arcs make a very *obtuse* angle with each other, in which case the actual *point* of intersection is not easily detected.

(9) Draw all lines *at first* as long as they are likely to be wanted, with the pencil, whether straight lines or circular arcs. For a straight line is not so accurately '*produced*' as it might in the first instance have been *continued* to the required extent. And it is best to continue circular arcs as far as they can be needed for two reasons: 1st, because, after once removing the compasses from the paper, you may miss the centre in the second attempt; and 2nd, because, if the compasses have been disturbed for any other purpose, or by accident, it may not be easy to hit the radius *to a nicety*.

(10) Never suppose, that you can '*produce*' a *very short* straight line, or draw another parallel to it with even tolerable accuracy, by means of a flat-ruler only. Nothing can be more fallacious, unless you have previously determined some distant point through which the line is to pass.

(11) It is not an easy thing even to *join* two points *accurately*, when the points are *very near* to each other. In many cases there will be some other longer line in the construction, to which the shorter line is known to be *parallel*. If so, apply the parallel-ruler to the longer line, as a test line, and then *force*, as it were, the shorter line into parallelism with this.

(12) In constructions where the same straight line or length, is repeated often, it is well to have two, and sometimes, three, pairs of compasses in use, in order that, being once adjusted to their proper distances, they may accurately retain that adjustment throughout. This is especially necessary in the construction of regular polygons.

(13) Despise not the *variety* of methods given for doing the same thing; because that which is practicable in one case is not so in another under different circumstances. Recollect, for example, that *theoretically* any straight line may be '*produced*'; but *practically* this is not true, because the line *may* already reach the edge of your paper, or your table, or room.

QUESTIONS AND EXERCISES E.

(1) What is meant, when we say, '*join AB*' and '*produce CD*'?

(2) When a straight line is required to be drawn very accurately between two given points on a plane surface mention the several things which ought to be carefully attended to.

(3) If a straight line is to be drawn to join two given points and to be produced, is there a greater chance of error when the points are near together, or when they are at a considerable distance apart? and why?

(4) *Exhibit* the probable error in 'producing' a *very short* straight line.

(5) What is the objection to determining a point in certain cases by the intersection of two straight lines? Exhibit an objectionable case. Which is the most favourable case?

(6) Is it possible to draw through a given point more than *one* straight line *parallel* to a given line? If not, why not?

(7) Why cannot a triangle have each of its angles just what we please?

(8) Is it absolutely *necessary* that the *two legs* of the Compasses should be precisely *equal?* There are two other conditions quite indispensable: what are they?

(9) In what way would you test the correctness of a *Parallel-Ruler?*

(10) If a square or rectangle be set before you, can you determine whether or not it be a *true* square or rectangle by means of the *Compasses alone?* If so, how?

(11) How should a Carpenter proceed to determine *how much* the end of a given board, professing to be rectangular, is *'out of square'*, when he knows that his *Square* is not trustworthy?

(12) In using the *Compasses* to draw a circle it is found that the circumference does not return into itself. State the two faults either of which may be the cause of this result.

(13) State the probable sources of error in drawing a circle by means of a *cord,* as described in (99).

(14) What is the objection to the Draughtsman's hasty mode of drawing a straight line, by means of the *'Triangle'* (102), at right angles to a given straight line *from a given point in the latter?* Is there the same objection to drawing a *perpendicular* by means of the same instrument?

(15) Through a given point draw the ***shortest*** straight line to meet a given straight line.

Through the same point draw *two equal* straight lines to meet the given line. Is there more than one solution of the last case?

(16) Bisect the base of an isosceles triangle by a straight line at right angles to it; and shew that the line will pass through the vertex of the opposite angle, and will also bisect that angle.

(17) Every point in the straight line which bisects an angle is *equidistant* from the two sides forming the angle. Hence, shew that in every triangle the straight lines which severally bisect the three angles meet in a point.

(18) Find a point equidistant from two given points. Is there more than one such point?

(19) Find a point equidistant from a given point, and a given straight line. Is there more than one such point?

(20) Construct an equilateral triangle of which the height only is given.

(21) When you say that one triangle is equal to another, do you mean that the *perimeters* of the triangles are equal, or their *Areas*?

(22) Can two *equilateral* triangles be equal, when their perimeters are unequal? Can two *isosceles* triangles be equal, when their perimeters are unequal?

(23) Construct an isosceles triangle of given perimeter upon a given base. When is this not possible?

(24) Through a given point on the ground lay down a straight line parallel to a given straight line in the same plane with a long cord and a few pegs only.

(25) Shew how to determine, *with the compasses alone*, whether any proposed quadrilateral figure be a *parallelogram* or not.

(26) If one straight line has been drawn to meet another straight line, and is said to be at right angles to it, how would you test the fact *with the compasses alone?*

(27) Shew how to determine, *with the compasses alone*, whether a proposed triangle be *right-angled* or not.

(28) In bisecting an angle explain why it is important to draw the intersecting arcs at some *considerable distance* from the vertex of the angle.

(29) From two given points draw two straight lines to meet in a given straight line and make equal angles with it.

(30) Change an equilateral triangle into a rectangle. Then make the rectangle into a *right-angled* triangle.

(31) Change a given square into two squares, 1st two *equal* squares, 2nd two *unequal* squares.

(32) Shew that the middle point of the hypothenuse of a right-angled triangle is always equidistant from the three angular points of the triangle.

(33) Construct a rectangle, having given its diagonal and one side.

(34) One angle of a parallelogram is *three-fourths* of a right angle; determine each of the other angles.

(35) Construct a lozenge of which both diagonals are given.

(36) Having given three different squares, make them all into *one* square.

(37) Having given the difference between the side of a square and its diagonal, construct the square.

(38) Draw a straight line of given length at right angles to the base, and terminated by one of the sides, of a given triangle.

(39) Can the angle at the base of an *isosceles* triangle be either equal to, or greater than, a right angle? If not, why not?

(40) Construct an isosceles triangle of which the base and opposite angle are given.

(41) A rectilineal figure of more than *three* sides may be equilateral without being equiangular; and *vice versâ*. Give examples.

(42) Shew that *every* straight line, drawn through the intersection of the diagonals of a parallelogram, and terminated by opposite sides, divides the parallelogram into two equal parts.

(43) A gate is to be strengthened by two rods, proceeding from the extremities of the lower horizontal bar, and meeting in the upper one. Shew that the least material will be used, when the rods are equal.

(44) P and Q are two given points without the given straight line AB. Find the shortest route from P to Q, upon condition of passing through some point in AB.

(45) Of all triangles having the same base and perimeter, shew that the isosceles is the *greatest*.

(46) Shew that *three* rods, whose lengths and relative position are given, cannot possibly form more than *one* triangle; whereas *four* rods, under like circumstances, will form an endless variety of *quadrilateral* figures. What conclusion do you draw from these facts, as to the proper construction of gates, roofs, &c.?

(47) Two pairs of parallel straight lines in the same plane intersect each other, draw another straight line intersecting both pairs, so that each pair shall intercept an equal portion of it. And give *two* solutions of the problem.

(48) From a given point draw a straight line to meet two given parallel straight lines in the same plane such, that the difference between the part intercepted by the parallels and the other part shall be equal to a given line.

(49) Draw the *shortest* straight line from the circumference of one given circle to that of another.

(50) Through a given point within a given circle draw the shortest chord.

(51) Shew how the diameter and radius of a given circle may be found by the *Tape* alone, without finding the centre.

(52) Draw a circle whose circumference shall cut a given straight line in two given points. Is there more than one such circle?

(53) From a given point within a given circle draw the shortest line to the circumference.

(54) Through a given point within a circle draw the chord which is bisected by that point.

(55) If two chords of a circle be given both in position and magnitude, describe the circle.

(56) Describe a circle which shall pass through a given point and touch a given circle at a given point.

(57) Find the point without a given circle, from which if a tangent be drawn to the circle, it will be equal to a given line.

(58) A carpenter has drawn the arc of a semicircle and wishes to verify it, but he has lost the centre; shew how he may do it by means of his *square* only.

(59) Draw the *diameter* of a given circle with the *square* alone without using the compasses.

(60) Divide the circumference of a circle into three equal parts *at one trial* by the compasses alone.

(61) Shew that the circumference of a circle is greater than *three*, and less than *four*, times its diameter.

(62) Trisect a given circle by straight lines drawn from the centre. Find also the sector which is the exact *twelfth* part of the circle.

(63) Compare the diameters of the circles *inscribed* in, and *circumscribed* about, the same equilateral triangle.

(64) Compare the areas of the squares *inscribed* in, and *circumscribed* about, a given circle.

(65) Shew that every parallelogram inscribed in a circle is a rectangle.

(66) Shew that the diameter of the circle *inscribed* in a *right-angled* triangle is equal to the excess of the sum of the two sides above the hypothenuse.

(67) Why is the hexagon the most easily constructed of all regular polygons?

(68) Having given the sides of a regular pentagon inscribed in a circle, shew how a regular polygon of *twenty* sides may readily be inscribed in the same circle.

(69) If a semicircle be described upon a side of a regular hexagon, and the adjacent side be produced to meet the circumference, shew that the *chord* thus formed is the side of another regular hexagon whose area is *one-fourth* of the former.

(70) How does it follow from the fact of a regular pentagon being *inscribed in a circle*, that each of the angles is equal to *three-fifths* of two right angles?

(71) Shew at once, *in the same manner*, that each angle of a regular hexagon is equal to *two-thirds*, and of a regular octagon to *three-fourths*, of two right angles.

(72) Shew that no triangle can be cut out of a square greater than half the square.

(73) Cut off the half of a given triangle by a straight line parallel to one of its sides.

(74) Construct a parallelogram which shall be equal to a given triangle both in area and perimeter.

(75) Make an equilateral triangle which shall be equal to the sum of two given equilateral triangles.

(76) Describe a circle whose circumference shall be to that of a given circle in a given ratio.

(77) Make a circle equal to half a given circle.

(78) Make a circle equal to the sum of *three* given circles.

(79) Find the square which is equal to a given hexagon.

(80) Divide a given square into three parts in the ratios 2, 3, 5, by straight lines drawn from one of the angular points.

(81) Divide a given triangle into three parts in the ratios 1, 2, 3, reckoning from one of the angular points, by straight lines parallel to the opposite side.

(82) Construct a square which shall be equal to a given rectangle and a given triangle taken together.

(83) If the points A, B, C, D, P be so situated that $PA : PB :: PC : PD$, shew that a circle may be drawn to pass through the points A, B, C, D.

(84) A straight line is divided into two given parts, find the point without it at which these parts shall subtend equal angles.

(85) Draw a straight line through two given concentric circles so that the two *chords* intercepted shall be in a given ratio.

(86) Divide a given triangle into three equal parts by straight lines at right angles to the base.

(87) Divide a quadrilateral figure, not a parallelogram, into three equal parts by straight lines at right angles to one of its sides.

(88) Divide a square into *five* equal parts by straight lines drawn from the intersection of its diagonals.

(89) From one of the angular points of a given triangle draw a straight line which shall divide the triangle into two such parts that one exceeds the other by a given smaller triangle.

(90) Shew that the perimeter of a square is *less* than that of any other parallelogram of equal area.

(91) Three given straight lines converge to one point, not determined, draw another straight line to meet them, such that the parts of it intercepted between each contiguous two are equal to one another.

(92) From a given point draw an arc of a circle which shall meet another *given* arc *tangentially*.

(93) Describe an oval whose extreme length shall be exactly twice its breadth.

(94) Draw a *two-centred* pointed arch of *given span and rise*.

(95) Draw a *two-centred* spiral which shall commence at one given point and terminate at another given point, after *one* complete revolution.

(96) Draw a *four-centred* spiral with the same conditions as in the last case.

END OF PART II.

ADVERTISEMENT TO PART III.

After much delay, arising from frequent ill health, I am at length enabled to bring this Part of my original design to completion, having obtained the valuable assistance of a kind friend, the Rev. F. Calder*, M.A. Head Master of the Grammar School, Chesterfield, to whom I am indebted for a large proportion of the *Exercises*, and for so much useful matter besides, that the book might fitly be called *Lund and Calder's Mensuration*.

It will be seen, at a glance, that this is not a work fashioned after the old pattern of English books on Mensuration; but it is grounded upon, and recognizes as a necessity, the Geometry of Euclid—that is, *Geometry as a Science*. It is constructed strictly on the deductive principle; and yet not so formally, as to exclude practical illustration, whenever it appeared desirable to introduce it. The Student's memory is not intended to be burdened with mere *Rules*, but each process is fully *reasoned* out; and the *units* of measurement, especially, are laid down and explained with extreme care, as being, in my opinion, the necessary first step to any sound and useful knowledge of the subject. I am persuaded, that very

* Author of a Treatise on *Arithmetic*, which has deservedly attained a high place in public estimation.

much of the failure of School-Mensuration, as hitherto taught, when it is brought to the test of the workshop, or the field, is to be ascribed to the vague notions, which young Students mostly obtain, of the *Units of Measurement*. This defect of former works has, therefore, been steadily kept in view; and I venture to hope it may be found, that the more *scientific* mode of treatment is after all the more truly *practical*.

This Part has so far exceeded the limits originally designed for it, that Part IV., *Geometry combined with Algebra* (if I am permitted to complete it), will now appear as a distinct work.

T. L.

MORTON RECTORY, ALFRETON,
Jan. 1, 1859.

CONTENTS.

Junior Students, on *first* reading this work, are recommended to pass over pages 244, 248 to 252, Arts. 243, 264, 273, 274, 276, 277, and 279.

CORRECTIONS.

PAGE		FOR	READ
203	Line 8 from bottom	*AC*	*AE.*
—	— 6, 5, and 4...	*ABCD*	*ABCE.*
213	Ex. 29, (6) Ans.	$\frac{149}{200}$	$\frac{29}{40}$.
215	Line 14 and 16	309760	3097600.
221	Ex. 12, Ans. (1)	$1\frac{7}{144}$	19.
—	— Ans. (2)	$\frac{5}{8}$	$\frac{1}{8}$.
—	Ex. 16, each of the three Answers should be halved.		
—	Ex. 18, Ans.	·5512	·552.
222	Ex. 27, Ans. (1)	6·364	2·1213...
—	Ex. 28, Ans.	1·183	1·1183.
223	Ex. 30, Ans.	8*s.* 3*d.*	16*s.* 6*d.*
251	Line 18	their chords	the No. of equal chords they contain.
298	Line 11	$CB = CD$	$CB = (\sqrt{3}-1) \times CD$, and $\therefore$ $PD = CD = CB \div (\sqrt{3}-1)$, which can be measured.
302	Ex. 18, Ans. after 'equal to' insert $\sqrt{\qquad\qquad}$.		
303	Ex. 22, Ans. (1)	$1003\frac{1}{2}$	$853\frac{1}{2}$.
—	— Ans. (2)	1r. 5·9p.	5·1744p.
319	Ex. 1, Ans.	sq. in.	sq. ft.
320	Ex. 5, Ans. (2)	11·03	11·48.
—	Ex. 8, Ans. (1)	44	$25\frac{1}{7}$.
—	— Ans. (2)	$49\frac{1}{2}$	$30\frac{25}{56}$.
353	Ex. 79, Ans.	97·03	98·01.
354	Ex. 88, Ans.	$\cdot0001\frac{4}{7}$	$\cdot0003\frac{1}{7}$.
355	Ex. 93, Ans.	4·33	4·165.
—	Ex. 99, Ans.	$\frac{23}{43}$	$\frac{97}{127}$.

ELEMENTS OF

GEOMETRY AND MENSURATION.

PART III.

GEOMETRY COMBINED WITH ARITHMETIC.
(MENSURATION.)

THE application of *Arithmetic* to Geometry enables us to perform various calculations and measurements with respect to *lines, lengths, distances, areas, angles,* &c., and is commonly called MENSURATION, that is, the Art and Science of *Measuring*.

GENERAL PRINCIPLES OF MEASUREMENT.

I. OF LINES.

214. The *Arithmetical Measure* of a *line* or *length* is the *ratio* which that *line*, or *length,* bears to another *line* or *length* taken for the *unit* or *standard* of measure.

Thus, if a certain *line*, or *length*, be called a *foot,* then another *line,* or *length,* which contains the former exactly *three* times, that is, which is just three times as long, will be 3 *feet.* And if 1 represent the former, 3 will represent the latter, line. Also another line which contains *the same unit five-and-a-half* times, will be represented by $5\frac{1}{2}$; and so on.

215. It is not *necessary* that the *unit* of lineal measurement be the *line*, or *length*, which we call a *foot;* it may be any other *line,* or *length*, taken at pleasure. But it *is* necessary, when *numbers* are used to represent Geometrical magnitude, in every case to know, and to bear in mind, *what unit*, or *standard,* has been employed.

Thus, if a certain *line*, or *length*, be represented by 13, we know nothing about it until we know what the *unit* is. If the *unit* be an *inch*, then the given line is 13 *inches;* or, if the *unit* be a *mile*, then the given line, or length, is 13 *miles:* and so on.

216. Hence, it is obvious, that much practical benefit arises from using those *units*, or *standards*, only which are *well known*, as an *inch*, a *foot*, a *yard*, a *mile*, &c. For thus we are enabled to communicate to others, by means of a few words and symbols, a true notion of the magnitude of any *line*, *length*, or *distance*, with which we are concerned, when it is once known to ourselves.

Thus, if we wish to inform a friend that we have walked a long distance within a certain time, by stating that we walked 100 *miles* in 3 *days*, we give him an accurate notion of our achievement, because a *mile* and a *day* are *units*, the one of *length*, and the other of *time*, with which he is supposed to be well acquainted.

217. It might, indeed, without the use of either Arithmetic or Geometry, be said that it is a *long* distance from London to Edinburgh. But this, in fact, expresses nothing which has any significance, because another person might as truly assert, that the same distance is *short*. Both may be right, having taken *different units* of measurement. For the distance between London and Edinburgh *is great* compared with the length of a man's *foot*, that is, when the *unit* is a *foot;* but the same distance is *small* when compared with the circumference of the Earth, or the distance to the Moon.

And so, then, let it be borne in mind, that the measuring of a *line*, or *length*, is simply the *comparing* it with some other *line*, or *length*, taken as a *standard;* and any proposed *line*, or *length*, is *great* or *small* only in respect of some other *line*, or *length*, with which the former is compared.

218. *To measure a given* straight *line or length.*

(1) When the given straight line is accessible in every part of it, let it be represented by *AB*, where *A* and *B* denote the extreme points. Take a foot-rule or yard-wand,

or some other convenient *standard* of measure, and lay it along AB, so as to have one of its ends coinciding with A. Mark where the other end meets the line, or length; from that point repeat the operation, as was done from A; and so on, until the *standard* has been laid along the whole line from A to B. This process enables us to *see* and *count* how many times the standard, or unit of measure, is contained in the given line; and that *number* of times is the *measure* of the line; for it is the *ratio* which the given line bears to the *unit* of measure.

This number may be either *whole* or *fractional*, according to circumstances. In some cases the *unit* will be contained an exact *integral* number of times in the given line; in others so many times, and *parts* of a time. And, in order that the *fractional* part may be readily determined, the *standard*, or *unit*, is divided into a certain number of equal parts, (as the foot-rule into 12 equal parts, called inches,) and each of these *parts* again into a certain number of equal parts; and so on, to any required degree of minute subdivisions. So that, if, for example, AB contain the *foot-rule* 3 times and five-twelfths of another time, then the *measure* of AB is $3\frac{5}{12}$ feet, or 3 feet 5 inches.

All this is so obvious to the senses *in practice*, that it requires no further illustration.

(2) For *short* lines the *draughtsman* more commonly makes use of the *compasses*, opening the compasses so as to separate their feet to the exact distance AB, he then applies that distance to the face of his flat ruler, which is divided into inches and parts of an inch; and thus he measures the line AB by *noting* how many inches and parts of an inch, (or, if AB be less than an inch, how many parts of an inch,) the compasses embrace.

Or, he places the edge of the graduated ruler *along the line* itself, which is to be measured, and observes *at once* with how many divisions of the ruler the given line coincides, and so measures it. This is often the most expeditious method.

(3) Lastly, for *long* lines a *tape* is commonly used, which is divided into feet and inches, and wound on a reel. One end of the tape is held at one end of the

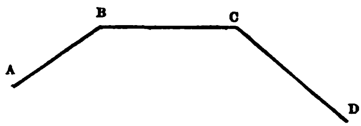
line, or length, to be measured, and the tape is then unwound, until, being tightly stretched, there is sufficient of it to cover the line in its whole extent. The figures marked on the tape, where it coincides with the other end of the line, express the length of the line in feet and inches.

Or, if the length to be measured be greater than the *whole length* of the tape, it is only necessary to repeat the operation by successive measurements, as in the first case.

The method of measuring still longer lines, or lengths, on the Earth's surface, as adopted by surveyors, will be given hereafter.

219. *To measure a given* crooked *line or length.*

(1) If the crooked line consist of two or more straight lines joined together, it is obvious that the whole line may be measured by adding together the measures of the several lines taken separately (determined as in the last Art.), which make up the whole.



Thus, it is evident that the measure of the crooked line $ABCD$ will be found by adding together the measures of AB, BC, and CD.

Or, with the *tape*, it may often be done in *one* single measurement. For, if $ABCD$ be the boundary of a rigid body, or if pegs be fixed at B and C, the tape may be tightly stretched so as to coincide with AB, BC, and CD, and thus shew *at once* the measure of $ABCD$.

(2) If the line, or length, to be measured be a *curved* line, its measure may be found by carefully laying a string upon it throughout its whole extent, and then applying the foot-rule, or other standard, to find the length of the string stretched out into a *straight* line.

Or, by means of a *tape*, a *curved* line, or length, may sometimes be measured *at one step*, since the tape combines in itself both the flexible string and the graduated measure. Thus, the woodman finds the girth of a tree,

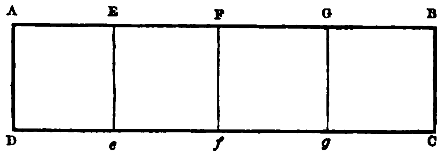
or the tailor the circumference of a man's body, in a moment of time.

II. OF SUPERFICIES, SURFACES, OR AREAS.

220. In the same manner as the *Arithmetical Measure* of a *line* is the *ratio* which that line bears to another *line,* taken as the *unit*, or *standard*—so the *Arithmetical Measure* of a *superficies*, *surface,* or *area* (all which mean the same thing), is the *ratio* which that *surface* bears to another *surface,* taken as the *unit,* or *standard,* of superficial measure. And as the *lineal foot* was stated to be often the most convenient *unit* of length for measuring *lines,* so the *square foot* (that is, the *square** of which each side is a lineal foot) is a common and convenient *unit*, or *standard*, of superficial measurement.

Hence, taking this *unit,* the measure of a *surface*, or *area,* is the *number* of times which that *surface*, or *area*, contains a *square foot;* and that number will be sometimes a *whole* number, and sometimes *fractional.*

For example, suppose the annexed diagram, *ABCD,*

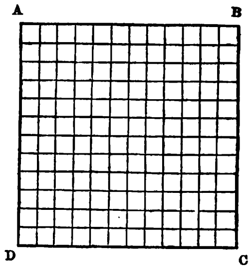


to be a miniature representation of a *rectangle,* of which the side *AB* is 4 feet, and the side *AD* is 1 foot; then, dividing *AB* into four equal parts (168, Part II.) in *E*, *F*, *G*, and drawing *Ee*, *Ff*, *Gg*, parallel to *AD,* or *BC*, it is obvious that we have divided the *rectangle* into 4 equal *squares*, each of which is a *square foot;* therefore the rectangle *ABCD* is plainly equal to 4 *square feet*, that is, the *measure* of the rectangle is 4, when the *unit* is a *square foot.*

221. But as it often happens, that a given *superficies, surface,* or *area,* which it is proposed to measure, does not contain an exact *integral* number of square

* It must be borne in mind that a *square* is not four straight *lines* of equal length and at right angles to each other, but the *plane area* included within those *lines.*

feet, therefore, just as the *lineal foot*, for a similar purpose, is divided into *inches*, so the *square foot* is divided into *square inches*. For instance, suppose the square $ABCD$ to represent a *square foot*. Divide AB, and AD, each into 12 equal parts (168, Part II.); these equal parts will be *inches*, since

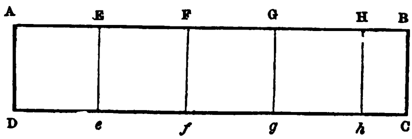


$AB = AD = 1$ foot$=12$ inches.

From the several points of division in AB draw lines through the square parallel to AD; and from the several points of division in AD draw lines parallel to AB. These lines will obviously divide the square $ABCD$ into 12 smaller squares repeated 12 times, that is, into 144 smaller squares; and each of these smaller squares is a *square inch*, that is, a square whose *side* is an *inch* in measure.

Hence, if a proposed surface do not contain the *square foot* an exact integral number of times, it will contain a certain number of *square feet*, and a fraction of a square foot over, which fraction may be a certain number of *square inches*. And then the *measure* of the surface, or area, will be so many *square feet* together with so many *square inches*.

For example, suppose $ABCD$ to be a rectangle, of

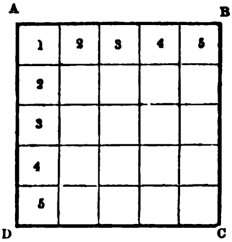


which the side AB is $4\frac{1}{2}$ feet, or 4 ft. 6 in., and AD *one* foot. Let $AE = EF = FG = GH = 1$ foot; and through E, F, G, H draw Ee, Ff, Gg, Hh parallel to AD or BC. Then $HB =$ half a foot, or 6 inches; and $HBCh =$ half a *square* foot, or 72 *square inches;* therefore the *measure* of the rectangle $ABCD$ is $4\frac{1}{2}$ *square feet*, or 4 *square feet* and 72 *square inches*.

If we still find, that the proposed surface is not made up of an exact integral number of *square feet and inches*, but that a *fraction* of a *square inch* remains over, then the *square inch* must be subdivided into smaller squares, as the *square foot* was; and so on, until we either obtain the precise measure of the proposed surface, or approach so near to it, that the remainder is of no account for practical purposes.

222. *To measure a given* square.*

1st. Let *ABCD* be the given square, of which each *side* is a certain *integral* number of units, 5 suppose. Divide *AB*, and *AD*, each into 5 equal parts (168, Part II.), and through the several points of division draw lines parallel to the sides of the square, dividing the given square, as in the diagram, into a certain number of equal smaller squares, each of which is the *unit* of superficial measure. The only question then is, what is the *number* of such squares contained in the proposed square? That *number* is the measure required; and is in this case obviously 5 repeated 5 times, that is, 25.



Thus, if *AB* be 5 *feet* in length, the area of the square *ABCD* will be 5×5, or 25, *square feet*. Or, if *AB* be 5 *yards* in length, then the square *ABCD* will be 25 *square yards*, that is, equal to 25 squares, each of which has 1 yard for its side.

The same rule will evidently hold whatever be the number of units in *AB*, that is, we must *multiply that number by itself to find the Arithmetical measure of the square ABCD.*

2nd. If the number of units in *AB* be not an *integer*, but *fractional*, as $5\frac{1}{4}$, each *unit* being reckoned in *quarters*, it is plain that each side of the square will contain 21 such *quarters*. Divide, then, *AB* and *AD*, each into

* *Given*, that is, as to the length of each of its *sides*. If the *side* be *not* given, but the surface, in the form of a square, be simply presented before us, then a *side* must be *measured* by Art. 218.

21 equal parts (168, Part II.), and through the several points of division draw lines parallel to AD and AB, as before, dividing $ABCD$ into equal smaller squares, each of which has a quarter of a unit for its side. The *number* of these smaller squares will obviously be 21 taken 21 times, or 21×21. But, as the *side* of the smaller square is $\frac{1}{4}$ of the lineal unit, the *square unit* contains 16 such squares. Therefore $ABCD$ contains $\frac{21 \times 21}{16}$ *square units;* that is, $ABCD$ is *measured* by $\frac{21 \times 21}{16}$, or $\frac{21}{4} \times \frac{21}{4}$, or $5\frac{1}{4} \times 5\frac{1}{4}$, the product of the side multiplied by itself, as before.

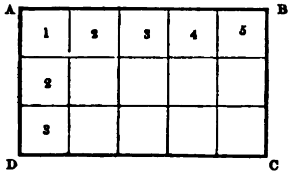
The same rule may be shewn to hold whatever fraction represents the side of the square, by dividing the lineal unit into as many equal parts as is expressed by the *denominator* of the fraction. Thus if AB be $3\frac{5}{8}$, each unit must be divided into *eighths*, that is, AB must be divided in 29 equal parts, and then, as before, it will readily be shewn that $ABCD$ is measured by $\frac{29}{8} \times \frac{29}{8}$, or $3\frac{5}{8} \times 3\frac{5}{8}$.

N.B. The *square*, of which any line, as AB, is a *side*, is commonly called the square *of* AB; and, for shortness, is written AB^2, and *read* '*AB square.*' Thus $AB^2 + OD^2$, which is *read* '*AB square plus OD square,*' means that the square upon OD is to be added to the square upon AB. Hence 5^2 does not stand for 5×5, that is, 5 times 5, *by definition merely*, but *is proved to be equal to* that product: in other words, *it is proved,* that the square, whose side is 5 *linear* units, contains 25 *square* units. And so also, whatever the number may be, which measures the *side* of a square, the *square* itself is measured by that number multiplied by itself.

Again, the half of the line AB is often written thus, $\frac{1}{2}AB$; and its square, as will readily appear by drawing the diagram, is *one-fourth* of the square of AB, which is written $\frac{1}{4}AB^2$. Similarly the square of $\frac{1}{4}AB$ is $\frac{1}{16}AB^2$; and so on—the *fraction* being multiplied by itself in all cases to obtain the square.

223. *To measure a given* rectangle.*

1st. Let *ABCD* be the given rectangle; and suppose *AB* to contain the lineal unit 5 times, and *AD* to contain it 3 times. Divide *AB* into 5 equal parts, and *AD* into 3 equal parts; then each of these parts is the lineal unit; and, if through the several points of division parallels to the sides of the rectangle be drawn, *ABCD* will obviously be cut up into a set of small *squares*, each of which is the superficial unit of measure (since its side is the lineal unit), and the number of these squares is plainly equal to 5 taken 3 times, or 3×5, the product of *AD* and *AB*.



The same will hold whatever other whole numbers of units are contained in *AB*, and *AD*, that is, *the measure of the rectangle ABCD is the product of AB and AD.*

Thus, if *AB* be 5 *feet*, and *AD* 3 *feet*, then *ABCD* will be 15 *square feet*. Or, if the lineal unit be a *yard*, then the measure of the same rectangle will be 15 *square yards*—the unit of *superficial* measure being always the square which has the lineal unit for its *side*.

2nd. If the *lineal* unit is not contained an exact integral number of times in either or both of the sides, the process of measurement is the same as that employed in the 2nd case of Art. 222. Thus, suppose the lineal unit to be contained $5\frac{1}{4}$ times in *AB*, and $3\frac{1}{2}$ times in *AD*; then *AB* being divided into 21 equal parts, and *AD* into 14 equal parts (each lineal *unit* being 4 such *parts*), and the parallels being drawn through the several points of division, the whole rectangle *ABCD* will be cut up into a set of small *squares*, each of which is the 16th part of the *square unit*, and the *number* of the *small squares* will plainly be 21 taken 14 times, or 14×21. Therefore the *measure* of the rectangle will be the 16th part of this

* Given, that is, as to the length of each of two adjacent *sides*. If the *sides* be not given, but the rectangle, as a surface, simply stands before us, then each of the two sides must be *measured* by Art. 218.

number, that is, $\frac{14 \times 21}{16}$, or $\frac{14}{4} \times \frac{21}{4}$, or $3\frac{1}{2} \times 5\frac{1}{4}$, the product of the two adjacent sides, as before.

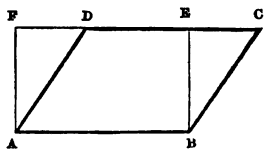
The same rule may be shewn, in a similar manner, to hold, whatever be the fractions which measure the sides of the rectangle. Hence we conclude that in all cases *the measure of a rectangle is the product of two adjacent sides.*

N.B. The adjacent sides, of which the product is taken, must be measured according to *the same unit*, before the multiplication takes place. And where some error of measurement is unavoidable, it is to be observed that the greatest care will be required in measuring *the lesser* of the two sides, because the *error* will be *multiplied* oftener when it has to be multiplied by the number of units in the longer side, than it would be, if the *same amount* of error is found in the longer side and is to be multiplied by the shorter.

It is to be observed also, that, whereas in Parts I. and II. a rectangle, as $ABCD$, is always spoken of as the 'rectangle *contained* by AB and BC,' or more shortly 'the rectangle AB, BC,' it is now *proved* that it is *equal to the product of* AB and BC.

224. *To measure a given parallelogram.*

Let $ABCD$ be the given parallelogram. From B and A draw BE, and AF, perpendiculars to CD, and CD produced. Then it is easily shewn that the triangle AFD is equal in all respects to the triangle BEC (24, Part I.); and thus it follows that the parallelogram $ABCD$ is equal to the rectangle $ABEF$. But the measure of the *rectangle*



$$ABEF = AB \times BE \ (223),$$

therefore the measure of the *parallelogram*

$$ABCD = AB \times BE$$

$= \textit{the base} \times \textit{the height}$, as it is usually stated.

In other words *the area of a parallelogram is equal to the product of any one side and the perpendicular distance of that side from the opposite side.*

In any proposed case, therefore, although the sides be given, it will be further necessary to measure this perpendicular distance, commonly called *the height* of the parallelogram.

Thus, if $AB = 5$ feet, and by measurement BE is found to be 3 feet, the area of the parallelogram $ABCD$ will be 3×5, or 15, square feet.

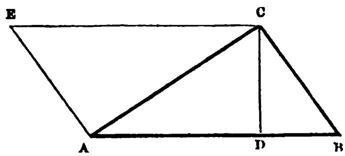
And, similarly, also, whatever be the numbers, whole or fractional, which measure AB and BE,

$$\text{the parallelogram } ABCD = AB \times BE.$$

Obs. It is plain that we could not cover the surface of the parallelogram $ABCD$, as represented in the diagram, by *any number* of *squares, however small,* on account of the acute and obtuse angles of the figure; but by the above device of converting the parallelogram into a *rectangle having the same area,* we are enabled to cover the surface of the latter with *square* units, and so to find the exact measure *in square units* of the equivalent *parallelogram.* This process may very fitly be called '*squaring the parallelogram.*'

225. *To measure a given triangle.*

Let ABC be the given triangle. The thing to be done is to find the equivalent *rectangle,* so that it may be possible to cover it with equal *squares,* and to find the *number* of those squares.



Through C draw CE parallel to AB, and CD perpendicular to AB; and through A draw AC parallel to BC.

Then $ABCD$ is a *parallelogram,* and the triangle ABC is *half* the parallelogram $ABCD$ (40, Part I.). But, by Art. 224, the parallelogram $ABCD$ is equal to the rectangle AB, CD, and is *measured* by $AB \times CD$; therefore the *triangle* ABC is measured by $\frac{1}{2}AB \times CD$, that is, *half the product of the base and height.*

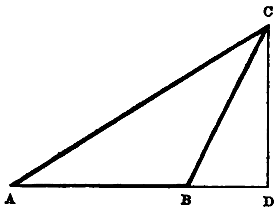
Thus, if $AB=7$ feet, and $CD=4$ feet, the triangle $ABC=\frac{1}{2}\times 7\times 4=14$ square feet.

To measure, therefore, a proposed triangle, measure *any one* of its sides, and the distance of that side from the vertex of the opposite angle; then *half the product of these two lengths will be the area required.*

But, observe, by distance of a side from the vertex of the opposite angle is meant the *shortest* distance, that is, the *perpendicular* let fall from the vertex to the side. And this perpendicular will in certain cases fall not upon the side itself, but the side *produced.* Thus in the annexed fig. the area of the triangle ABC is equal to

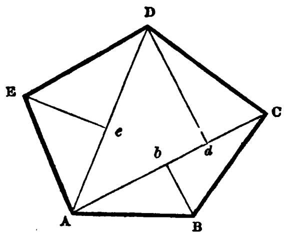
$$\tfrac{1}{2}\times AB\times CD,$$

where CD is the perpendicular from C upon AB *produced.*



226. *To measure a given* rectilineal *surface of any number of sides.*

Let $ABCDE$ be the proposed surface to be measured (the process is the same whatever be the number of the sides). From the vertex of any *one* of the angles, as A, draw the diagonals AC, AD, so as to divide the whole surface into the triangles ABC, ACD, ADE. Then the surface $ABCDE$ is manifestly equal to the sum of the three triangles, each of which may be measured separately by (225).



Thus, drawing the perpendiculars Bb on AC, Dd on AC, and Ee on AD, by Art. 225,

$$\text{Area of triangle } ABC=\tfrac{1}{2}AC\times Bb,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots \; ACD=\tfrac{1}{2}AC\times Dd,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots \; ADE=\tfrac{1}{2}AD\times Ee;$$

therefore area of

$$ABCDE=\tfrac{1}{2}AC\times Bb+\tfrac{1}{2}AC\times Dd+\tfrac{1}{2}AD\times Ee;$$

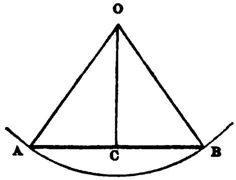
and by measuring the *lines* AC, AD, Bb, Dd, and Ee, the measure of the surface required is known.

[This Problem will be more fully discussed in the section on SURVEYING.]

227. *To measure the perimeter and area of a given* regular Polygon *of any number of sides.*

This is obviously only a particular case of the preceding problem; but as it furnishes a *general* result applicable to *all regular* polygons whatever, it may be fitly inserted here.

Let AB be one of the sides of a regular polygon; O the centre of the circumscribing circle (162, Part II.); draw OC perpendicular to AB, and join OA, OB.



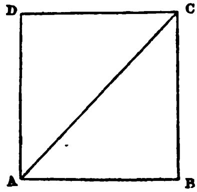
(1) Then, to obtain the measure of the *perimeter*, it is plain that we have only to measure AB, and multiply it by the number of sides in the polygon.

(2) Also, the *area* of the polygon will be equal to the area of the triangle OAB multiplied by the number of sides in the polygon, that is, $\frac{1}{2}AB \times OC \times$ number of sides, or $\frac{1}{2}OC \times$ *perimeter*.

OBS. It is to be observed that the proposition proved in (43, Part I.), viz. that in any right-angled triangle the square of the hypothenuse is equal to the *sum* of the squares of the sides bounding the right angle, is of continual application in Mensuration, and enables us to measure squares, rectangles, and triangles, without the precise data supposed in the preceding Articles. Thus,

228. *To measure a square when the diagonal only is given.*

Let $ABCD$ be the square, of which the diagonal AC is known: then, by (43),



$$AC^2 = AB^2 + BC^2$$
$$= \text{twice } AB^2, \quad \because AB = BC,$$
$$\therefore AB^2 = \tfrac{1}{2}AC^2;$$

or $ABCD$ = half the square of AC.

If, for example, $AC=3$ in.

then $ABCD=\frac{9}{2}=4\frac{1}{2}$ square in.

By the same reasoning also it appears that the diagonal of a *square* bears an invariable ratio to the side. For

$$AC^2 : AB^2 :: 2 : 1,$$

$$\therefore\ AC : AB :: \sqrt{2} : 1.$$

Hence, the *diagonal* of a square is found from the *side* by multiplying the latter by $\sqrt{2}$; and the *side* from the *diagonal* by multiplying half the latter by $\sqrt{2}$. For by (74, Part I.)

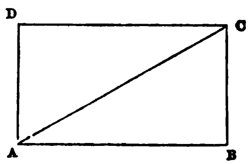
$$AC=\sqrt{2}\times AB;\ \text{and}\ AB=\frac{AC}{\sqrt{2}}=\tfrac{1}{2}\sqrt{2}\,.\,AC.$$

Since $\sqrt{2}=1{\cdot}414213\ldots$ *in infinitum*, the ratio of the diagonal of a square to its side cannot be expressed *accurately*, but we can approach as near as we please to accuracy by taking a sufficient number of decimal places. For many purposes it will be accurate enough to write $1\frac{4}{10}$, or $\frac{7}{5}$, for $\sqrt{2}$.

Again, since the ratio $\sqrt{2} : 1$ cannot be accurately expressed, this shews that *the diagonal of a square and its side are incommensurable,* which means that, although *each* can be measured accurately by *some* unit, they cannot *both* be measured by the *same* unit, however small that unit may be.

229. *To measure a rectangle, when the diagonal and one of the sides are given.*

Let $ABCD$ be the rectangle, of which the diagonal AC, and the side AB are given.



Then, by (223),

Area of $ABCD=AB\times BC$,

But $BC^2=AC^2-AB^2$, (43),

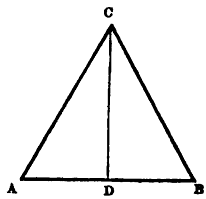
$\therefore\ ABCD=AB\times\sqrt{AC^2-AB^2}$.

If $AC=5$ feet, and $AB=4$ feet,

then, area $ABCD=4\times\sqrt{25-16}=12$ square feet.

230. *To measure a triangle when the* sides *only are given.*

Let ABC be the triangle, CD a perpendicular from C upon AB, *supposed,* but not known. And



1st. Let the triangle be *equilateral;* then it is easily proved that

$$AD = BD = \tfrac{1}{2}AB.$$

And area of $ABC = \frac{1}{2}AB \times CD$, (225)

$$= \tfrac{1}{2}AB \times \sqrt{AC^2 - AD^2}, \quad (43)$$
$$= \tfrac{1}{2}AB \times \sqrt{AB^2 - \tfrac{1}{4}AB^2},$$
$$= \tfrac{1}{2}AB \times \sqrt{\tfrac{3}{4}AB^2}.$$

If $AB = 4$ feet, then

area of triangle $= 2 \times \sqrt{12} = 4\sqrt{3}$ square feet.

2nd. Let the triangle be *isosceles,* but not equilateral.

Then area $= \frac{1}{2}AB \times CD$,

$$= \tfrac{1}{2}AB \times \sqrt{AC^2 - AD^2},$$
$$= \tfrac{1}{2}AB \times \sqrt{AC^2 - \tfrac{1}{4}AB^2}.$$

If $AB = 4$, and $AC = BC = 10$, then

area of triangle $= 2\sqrt{96} = 8\sqrt{6}$.

3rd. Let the triangle have all its sides *unequal;* and the length of each side known;—

In this case the area of the triangle can be found from these data alone, but only by means of *Algebra.* The algebraical process, however, furnishes the following general Rule, which is of easy application:—

RULE. *Add together the numbers (expressed in the same unit) which measure the three sides, and take half their sum. Let the result be represented by* S; *then subtract each side separately from* S, *and find the continued product of* S, S−AB, S−AC, *and* S−BC. *The square root of that product will be the area of the triangle required.*

Ex. Let the sides of the triangle be 3, 6, 7, yards respectively; to find the area of the triangle.

$$3+6+7=16, \qquad \therefore S=8,$$
$$S-AB=8-3=5,$$
$$S-AC=8-6=2,$$
$$S-BC=8-7=1;$$

$\therefore$ area $=\sqrt{8\times5\times2\times1}=\sqrt{80}=8{\cdot}95$ square yards nearly.

231. When the length and breadth of a rectangular figure are given, we have seen (223) that the area is found by multiplying together these two dimensions, taking care that they are both measured by the same unit.

These dimensions are often expressed in feet; or in feet and inches; or in feet, inches, and fractional parts of an inch. If these fractional parts are halves, or quarters, it is usual to express them in *twelfths*. We subjoin examples.

1st. Let the dimensions be expressed solely in feet, as 5 ft. by 3 ft. Then the area is equal to 5×3, or 15 square feet.

2nd. Let the dimensions be in feet and inches; as 5 ft. 3 in. by 4 ft. 6 in.

Here it must be observed, that in the required multiplication, we have these results, viz.

1 foot × 1 foot = 1 sq. foot,

1 foot × 1 inch = an area of 12 sq. inches,

1 inch × 1 inch = 1 sq. inch.

Hence, in the subjoined operation the work is performed as in compound multiplication, and the units in each denomination are converted into the next higher by dividing by 12; so that the result is 23 sq. ft., 7 areas of 12 sq. inches and 6 sq. inches; or 23 sq. ft. 90 sq. inches.

$$\begin{array}{r} 5\,.\,3 \\ 4\,.\,6 \\ \hline 2\,.\,7\,.\,6 \\ 21\,.\,0 \\ \hline 23\,.\,7\,.\,6 \\ \hline \end{array}$$

3rd. Let the dimensions be in feet, inches, and twelfths of an inch; as in a rectangle 5 ft. 4 in. 3 twelfths, by 2 ft. 6 in. 9 twelfths.

We have now to observe that

$$1 \text{ foot} \times \tfrac{1}{12} \text{ inch} = 12 \text{ in.} \times \tfrac{1}{12} \text{ in.} = 1 \text{ sq. in.}$$
$$1 \text{ in.} \times \tfrac{1}{12} \text{ in.} = \tfrac{1}{12} \text{ sq. in.},$$
$$\tfrac{1}{12} \text{ in.} \times \tfrac{1}{12} \text{ in.} = \tfrac{1}{144} \text{ sq. in.}$$

Hence, in the accompanying operation, the product is obtained by the same mode as in the last example, and is 13 sq. ft., 8 areas of 12 sq. in., 7 sq. in., 8 areas each $\frac{1}{12}$ sq. in., and 3 each $\frac{1}{144}$ sq. in. The second and third products when reduced to one name amount to 103 sq. in.; and the fourth and fifth to $\frac{99}{144}$ of a square inch: or, the whole = 13 sq. ft. 103 $\frac{11}{16}$ sq. in.

```
            5 . 4 . 3
            2 · 6 . 9
        -------------
        4 . 0 . 2 . 3
    2 . 8 . 1 . 6
   10 . 8 . 6
   ------------------
   13 . 8 . 7 . 8 . 3
```

4th. If the fractional parts of an inch cannot be converted into *twelfths*, it is better to bring the whole to feet and decimal or fractional parts of a foot, or of an inch.

Thus 7 ft. $2\frac{1}{5}$ in. would be better converted into $7\frac{11}{60}$ ft., or $86\frac{1}{5}$ in., or 86·2 in. If the resulting decimal would be recurring, a vulgar fraction is preferable.

Ex. Find the rectangular area, whereof the length and breadth are 3 ft. $5\frac{1}{7}$ in., and 2 ft. $6\frac{2}{9}$ in. respectively.

The product $= 41\frac{1}{7} \times 30\frac{2}{9}$ sq. inches

$$= \frac{41\frac{1}{7} \times 30\frac{2}{9}}{144} \text{ sq. ft.}$$

$$= \frac{288}{7} \times \frac{272}{9} \times \frac{1}{144} = \frac{544}{63} \text{ sq. ft.}$$

$$= 8\tfrac{40}{63} \text{ sq. ft.}$$

If the dimensions be 1 ft. 10·6 in. and 0·275 in.

the product $= 22{\cdot}6 \times 0{\cdot}275$ sq. in.

$= 6{\cdot}215$ sq. in. $= {\cdot}04316$ sq. ft. nearly.

QUESTIONS AND EXERCISES F.

(1) What is meant by the *length* of a line in Mensuration?

(2) What is meant by choosing a *unit* of measurement?

(3) Shew by examples that there is an advantage in selecting particular *units* of measurement.

(4) How many different *kinds of units* can be employed in Mensuration? Are they all employed in the preceding section.

(5) If, in ascertaining the length of a line, according to Question (1), you find the result to be a *fraction,* how do you interpret that result?

(6) Describe the common modes of measuring (1) short straight lines, (2) long straight lines.

(7) Shew how to measure a proposed crooked, or curved, line.

(8) If one foot is taken as the *unit* of length, what are the numbers which represent the several lengths of 3 in., $2\frac{1}{4}$ in., 2 ft. 3 in., 7 yds., all in terms of that unit?

(9) What is the mode of ascertaining the number of units in any proposed *square?*

(10) What is the reverse process, viz. when the number of units in the *square* is given, to find the length of one *side?*

(11) Shew how the Table commonly called "Square Measure" is constructed.

(12) What is the most ready mode of finding the *area* of a triangle?

(13) Write down the expression for the area of a triangle, in terms of its base and perpendicular height; and shew that if the base of a triangle be ·625 poles and its height 1·2 poles, its area is 11·34375 sq. yds.

(14) Define a *rectangle* what other name is given to it?

(15) Can we always express the area of a proposed rectangular figure in terms of *units* arbitrarily taken; if not, why not?

(16) Describe the required measurements and calculations for finding the *area* of a *rectangle*.

(17) Suppose a parallelogram be not *rectangular*, what are the required measurements for finding its area?

(18) Give an example of two lines which are *incommensurable*, and shew that they are so.

(19) Deduce from the measurement of a parallelogram the measurements required for finding the area of a triangle.

(20) The area of a parallelogram is ·375 sq. yds.; its base is ·75 yds.; what is the perpendicular height?

Ans. $\frac{1}{2}$ yd.

(21) The area of a triangle is 3·525 acres, and the perpendicular from the vertex of one angle on the opposite side is 193·6 yds.; find the length of that side.

Ans. 176·25 yds.

(22) Deduce from question (19) the mode of measuring a plane surface bounded by any given number of straight lines.

(23) How may the perimeter and area of a regular polygon be ascertained?

(24) In the following irregular four-sided figures, given the diagonal, and the perpendiculars upon it from the opposite angles, in each case, find the areas.

	Diagonal.	Perps.
(1)	27·6 ft.	13·2 ft., and 11·7 ft.
(2)	35 ft. $2\frac{1}{2}$ in.	17 ft. 2 in., and 16 ft. 3 in.
(3)	100·26 ft.	35·7 ft., and 45·9 ft.

(1) Ans. 343·62 sq. ft. (2) Ans. 588 sq. ft. $39\frac{1}{4}$ sq. in
(3) Ans. 454 sq. yds. 4·608 sq. ft.

(25) In the following five-sided figures, given the diagonals and three perpendiculars from opposite angles, in each case, find the areas.

Note. The first two perpendiculars are upon the first diagonals.

	Diagonals.	Perps.
(1)	35; 40;	15; 12; 14.
(2)	16·7; 19·4;	8·2; 3·6; 5·9.
(3)	12·375; 18·12;	7·56; 8·2; 4·95.

(1) Ans. 752·5. (2) Ans. 155·76. (3) Ans. 142·362.

(26) The perimeter of a regular hexagon is ·75 ft.; find the radius of the circumscribing circle.

Ans. $1\frac{1}{2}$ inches.

(27) Find the areas corresponding to the following lengths and breadths of rectangular figures:—

	Length.	Breadth.
(1)	4 ft. 3 in.	3 ft. 6 in.
(2)	13 ft. 6 in.	12 ft. 9 in.
(3)	7 ft. $3\frac{1}{4}$ in.	8 ft. 6 in.
(4)	21 ft. $3\frac{1}{2}$ in.	17 ft. $5\frac{3}{4}$ in.

(1) Ans. $14\frac{7}{8}$ sq. ft. (2) Ans. 172 sq. ft. 18 sq. in.
(3) Ans. 61 sq. ft. $115\frac{1}{2}$ sq. in.
(4) Ans. 372 sq. ft. $23\frac{1}{8}$ sq. in.

(28) The following dimensions of rectangles are expressed in feet and decimal parts of a foot; or in inches and decimal parts of an inch. Find the areas.

	Length.	Breadth.
(1)	7·5 ft.	3·27 ft.
(2)	9·375 ft.	8·024 ft.
(3)	1·275 in.	·075 in.
(4)	3·304 ft.	7·85 in.

(1) Ans. 24·525 sq. ft. (2) Ans. 75 sq. ft. $32\frac{2}{5}$ sq. in.
(3) Ans. ·095625 sq. in.
(4) Ans. 2 sq. ft. 23·2368 sq. in.

(29) Find the areas of the triangles, whereof one side and the perpendicular thereon from the vertex of the opposite angle are respectively as follows:

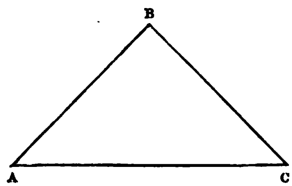
(1) 6 ft. 2 in. and 3 ft. $0\frac{1}{5}$ in.
(2) 3 ft. $9\frac{1}{4}$ in. ... 1 ft. $11\frac{1}{3}$ in.
(3) 100 ft. 10 in. ... 15 ft. 7·2 in.
(4) 16 ft. 8·5 in. ... 7 ft. 9·5 in.
(5) 13·25 in. ... 3·6 in.
(6) 10·275 ft. ... 3·875 ft.

(1) Ans. 9 sq. ft. 43·4 sq. in.
(2) Ans. 3 sq. ft. $95\frac{11}{12}$ sq. in.
(3) Ans. $786\frac{1}{2}$ sq. ft.
(4) Ans. 65 sq. ft. $13\frac{3}{8}$ sq. in.
(5) Ans. 23·85 sq. in.
(6) Ans. 19 sq. ft. $130\frac{148}{200}$ sq. in.

PROBLEMS.

The following easy Problems, fully worked out, will serve to direct the student in the practical application of the preceding propositions and principles.

PROB. 1. Two equal rafters are to be framed together, at right angles, to span a building of given width; find the length of each rafter.



Let AB, BC, be the equal rafters, the angle ABC a right angle, then, by (228),

$$AB=\frac{AC}{\sqrt{2}}, \text{ or } \frac{AC}{2}\sqrt{2}.$$

If AC=10 feet, since

$$\sqrt{2}=1{\cdot}4142\ldots\ldots$$

$$AB \text{ or } BC=5\times 1{\cdot}4142\ldots\ldots \text{ ft.}=7{\cdot}071\ldots\ldots \text{ ft.}$$

The object of multiplying $\dfrac{AC}{\sqrt{2}}$ by $\sqrt{2}$, as has been done here, is to avoid having to *divide* by the interminable divisor $\sqrt{2}$, or 1·4142...... For though it is very easy to *multiply* by such a number, yet to *divide* by it

involves a long process; and since the divisor is not exactly known, it is more difficult to estimate the *error* in this case, than in a process involving only multiplication.

If the sides AB and BC be not equal, yet either of the two may be found, when the magnitude of the other is given. For we have by (43)

$$AC^2=AB^2+BC^2, \therefore AB^2=AC^2-BC^2, \text{ and } BC^2=AC^2-AB^2.$$

Ex. Let $AC=10$, $BC=3$,

$$\text{then } AB^2=100-9=91;$$
$$\therefore AB=\sqrt{91}=9{\cdot}53, \text{ \&c.}$$

Prob. 2. A portion of the crooked fence of a field projecting outwards is in the form of two straight lines at right angles to each other, as AB, BC, in the last example, being 96 and 64 yds., respectively; find how much fencing will be saved, by taking it direct from A to C; and how much will the field be diminished in area?

$$AC=\sqrt{AB^2+BC^2}=\sqrt{9216+4096},$$
$$=\sqrt{13312 \text{ sq. yds.}}*,$$
$$=115{\cdot}4 \text{ yds. nearly};$$

and subtracting this from $96+64$, we have the saving of fence $=44{\cdot}6$ yds.

Also, the field is diminished by the area of the triangle ABC; and this area, by (225),

$$=\frac{AB\times BC}{2}=96\times 32 \text{ sq. yds.}=3072 \text{ sq. yds.}$$

Prob. 3. From the corner of a given square field it is required to fence off *half* of the square, in the form of a square, and to find the length of the fence.

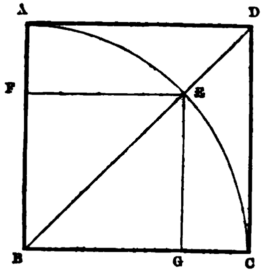
Let $ABCD$ be the square; join BD, and with centre B and radius BA describe the quadrant AEC, cutting BD in E; from E draw EF, EG, parallel to BC and AB respectively; $BFEG$ shall be the square required.

* It must be borne in mind, that, when the *square root* of any number of *square* yards, feet, &c., is extracted, the result is in *linear* yards, feet, &c., and *vice versâ*, when *linear* yards &c. are squared, the result is in *square* yards, &c.

For, by (228), the square $BFEG$=half the sq. of BE,

$=\frac{1}{2}AB^2$,

$=\frac{1}{2}ABCD$.



And length of fence required

$=FE+EG$,

$=$twice $BG=2\dfrac{BE}{\sqrt{2}}=BE\times\sqrt{2}$,

$=AB\times\sqrt{2}$.

PROB. 4. To find the number of acres in a square mile.

A mile =1760 yds. ;

$\therefore$ a square mile$=1760\times1760$ sq. yds. by (222),

$=3097600$ sq. yds.

Again, 1 acre$=4840$ sq. yds.;

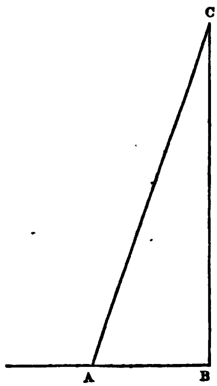
$\therefore$ number of acres in 1 sq. mile$=\dfrac{3097600}{4840}=640$.

PROB. 5. Find the length of the shortest ladder which will reach the eaves of a house 30 ft. high, if the foot of the ladder is to be placed at a distance of 10 feet from the house.

Let AC be the ladder, and BC the wall of the house;

$\therefore AC^2=(10)^2+(30)^2=1000$ sq. ft.;

$\therefore AC=\sqrt{1000 \text{ sq. ft.}}=31\cdot62$ ft.



Again, if the length of the ladder were given 30 ft., and height of the house 24 ft.; find the point at which the foot of the ladder must be placed, so as just to reach the eaves.

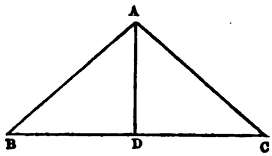
$AB^2=AC^2-BC^2$,

$=900-576$,

$=324$ sq. ft.;

$\therefore AB=18$ ft.

PROB. 6. A certain triangular court-yard is known to have two of its sides equal to one another, being 15 yds. each; and the third side is 24 yds. Find the cost of paving the court-yard at 15*d.* per square foot.



In the triangle ABC let

$AB=AC=15$ yds.,

and $BC=24$ yds.

Draw AD perpendicular to BC.

Then D is the middle point of AC.

Also, $AD^2=AB^2-BD^2=(15)^2-(12)^2$,

$=225-144=81$ sq. yds.;

$\therefore AD=9$ yds.

And the area of $ABC=\frac{1}{2}BC\times AD$, (225),

$=12\times 9$ sq. yds.,

$=12\times 9\times 9$ sq. ft,

And the cost of paving at 15*d.* per sq. ft. $\Big\}=12\times 9\times 9\times 15$ pence,

$$=\frac{12\times 9\times 9\times 15}{12\times 20}£=\frac{243}{4}=\overset{£.}{60}\ \overset{s.}{15}.$$

PROB. 7. A square picture is surrounded by a flat frame 3 inches broad. The area of the picture and frame together is $12\frac{1}{4}$ sq. ft. How much would it cost to varnish the picture at $\frac{1}{8}d.$ per sq. inch?

Length of a side of the frame

$=\sqrt{12\frac{1}{4}\text{ sq. ft.}}$, (222),

$$=\sqrt{\frac{49}{4}\text{ sq. ft}}=\frac{7}{2}\text{ lin. ft.}=42\text{ in.}$$

Subtracting *twice* the breadth of the frame, viz. 6 inches, we have the length of a side of the picture $=36$ inches, and the area to be varnished $=36\times 36$ sq. in.:

$\therefore$ the cost of varnishing at $\frac{1}{8}d.$ per sq. in.

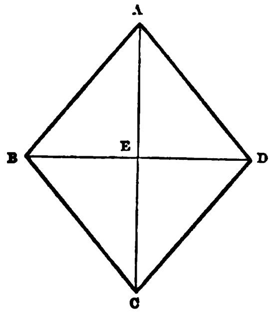
$$=36\times 36\times\tfrac{1}{8}d.$$

$$=\frac{36\times 36}{12\times 8}s.,$$

$$=\frac{27}{2}s.=13s.\ 6d.$$

Prob. 8. Find the cost of gilding an escutcheon, in the form of a lozenge, of which the diagonals are 12 inches and 9 inches, at 9*d.* per sq. in.

Let $ABCD$ be the lozenge (13),



its area $=2ABD$,

$=2AE\times BE$,

since AC and BD are at right angles (127),

$=\dfrac{AC\times BD}{2}$,

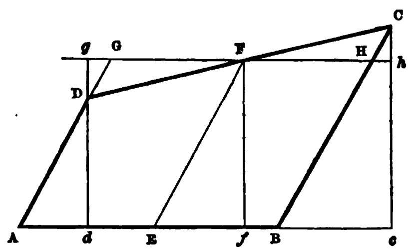
$=\dfrac{12\times 9}{2}$ sq. in.;

$\therefore$ the cost of gilding

$$=\frac{12\times 9}{2}\times\tfrac{3}{4}s.=\frac{81}{2}s.=£2.\ 0s.\ 6d.$$

Prob. 9. To find the area of a trapezium, *i.e.* a four-sided figure, of which two sides are parallel.

Let $ABCD$ be the proposed trapezium, of which the



sides AD and BC are parallel. Bisect AB in E, and DC in F, and join EF. Through F draw GFH parallel to AB, meeting BC in H, and AD produced in G. Then, since F is the middle point of DC, it is easily shewn that the triangle GFD is equal to the triangle FCH; hence, in measuring the trapezium $ABCD$, if the triangle FCH be cut off, and we take in its stead DFG, the result will be the same.

Through D, F, C, draw perpendiculars gd, Ff, Cc, to the base, and produce GH both ways to meet Dd and Cc in g and h.

Then the area of $ABCD$ = area of $ABHG = AB \times Ff$. Ff is the average height of the trapezium; hence the area = base × average height.

For $2Ff = dg + ch$,

$$= dD + Dg + ch,$$

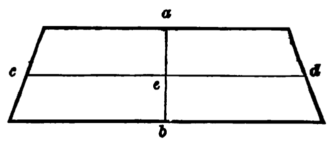
$$= dD + Cc, \text{ since } Dg = Ch;$$

$$\therefore\ Ff = \tfrac{1}{2}(Dd + Cc),$$

$= \frac{1}{2}$ the sum of the greatest and least heights.

If the $\angle ABC$ be a right angle, the greatest and least heights become the parallel sides; and then the area = base × $\frac{1}{2}$ sum of the parallel sides.

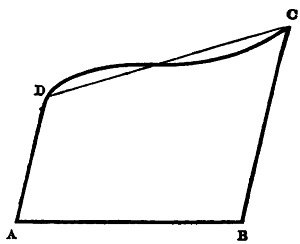
In the annexed particular example of a symmetrical trapezium, where ab bisects the parallel sides, and cd, meeting ab in e at right angles, bisects the other sides, we may consider the figure to consist of *two* trapeziums, having a common base ab, and average height ec, or ed; then the



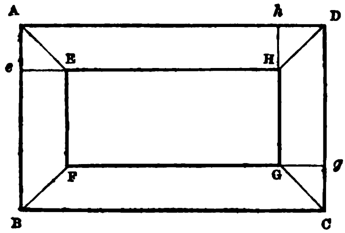
area = $ab \times cd$ = breadth × average length;

or = breadth × $\frac{1}{2}$ sum of the parallel sides.

NOTE. If DC were a curved line, a straight line might be so drawn, terminated in AD and BC, or in those lines produced, partly above and partly below the curve, that the included and excluded areas should nearly balance; then the area of the irregular surface will *approximately* be represented by that of the trapezium, found as before.



PROB. 10. Find the area of a flat oblong frame of uniform width, as of a picture, or a walk round a rectangular grass-plot.



Let $ABCD$ be the outer, and $EFGH$ the inner, boundary.

Produce

GH to meet AD in h,
FG ...... CD in g,
HE ...... AB in e.

The area required may be obtained in several ways.

1st. It is equal to

the area of the rectangle BD – area of rectangle FH,

$$= AB \times AD - EF \times EH.$$

2nd. It is equal to

twice the rectangle AH + twice the rectangle DG,

$$= 2Ah \times Hh + 2Gh \times Dh,$$
$$= 2(Ah + Gh) \times \text{breadth of frame.}$$

3rd. Or, by joining the corresponding corners, by the lines BF, CG, &c., the frame is divided into four trapeziums, of the form exhibited in the 2nd. case of Prob. 9. And since the area of each = breadth × $\frac{1}{2}$ sum of parallel sides,

$$\therefore \text{the area of the four} = \begin{cases} \text{breadth} \times \frac{1}{2} \text{sum of outer and inner} \\ \text{boundaries of the frame.} \end{cases}$$

EXERCISES G.

(1) Each side of a square is 75·4 feet; what is the distance between the two opposite corners? Ans. 106·63068 ft.

(2) A roof of 10 ft. span, and formed by two equal rafters makes a right angle at the ridge; find the length of the slope from the ridge to the eave. Ans. 7·071...ft.

(3) How many plots, each 11 yards square, can be obtained from 50 acres? Ans. 2000.

(4) The foot of a mast 120 ft. high is 25 ft. from the ship's side; find the length of a rope reaching from the side to a point in the mast at two-thirds of its height. Ans. 83·815...ft.

(5) A ten acre field is in the form of a square; find the cost of laying down a diagonal drain at 15*d*. per linear yard. Ans. £19. 8*s*. $10\frac{1}{4}$*d*.

(6) The diagonal of a square board is 10 yards; what is the side of the square, and its area?

(1) Ans. 7·071...yds. (2) Ans. 50 sq. yds.

(7) The sides of a rectangular plot are 108 ft. and 144 ft.; if the former dimension be shortened by 12 ft., how much must the latter be increased, so that the area may remain unaltered? Ans. 18 ft.

(8) In the preceding example, if the longer side be shortened by 12 ft., how much must the other be increased? Ans. $9\frac{9}{11}$ ft.

(9) Compute the lengths of the outer boundaries of each of the three rectangular plots in (7) and (8). Ans. 504; 516; $499\frac{7}{11}$.

(10) The sides of a triangle are 4, 5, and 6; alter the last dimension so that the triangle shall become right-angled. Ans. Add ·403...to it.

(11) Out of a piece of metal, 15 inches square, as many circular portions as possible are cut, each 1 inch in diameter; how many will there be? Ans. 225.

(12) Find the areas of the trapeziums of which the dimensions are as follows:

	Parallel sides.	Perp. distance.
(1)	1 ft. 3 in.; 1 ft. 7 in.	2 ft. 11 in.
(2)	3 ft. $2\frac{1}{2}$ in.; 4 ft. 9 in.	$27\frac{1}{2}$ in.
(3)	3·75 ft.; 4·95 ft.	1·5 ft.
(4)	54 yds.; 60 yds.	30 yds.
(5)	2·5 poles; 1·84 poles.	7 poles.
(6)	385 ft.; 450 ft.	125 ft.

Ans. (1) 4 sq. ft. $1\frac{7}{144}$ sq. in.
Ans. (2) 9 sq. ft. $17\frac{5}{8}$ sq. in.
Ans. (3) 6·525 sq. ft.
Ans. (4) 1710 sq. yds
Ans. (5) 15·19 sq. poles.
Ans. (6) 1 ac. $958\frac{11}{18}$ yds.

(13) The hypothenuse of a right-angled triangular field is 50 yds., and the other sides are in the ratio of 3 to 4; find its area, and cost at 80 guineas per acre.

(1) Ans. 600 sq. yds. (2) Ans. £10. 8*s.* $3\frac{21}{121}$*d.*

(14) Find the side of a square which cost £27. 1*s.* 6*d.* paving, at 8*d.* per square yard. Ans. $28\frac{1}{2}$ yds.

(15) How many square feet of flooring can be covered by a board whose length is 10 ft. 5 in. and the breadths of the two ends $2\frac{1}{2}$ ft. and $1\frac{3}{4}$ ft.?

Ans. 22 ft. $19\frac{1}{2}$ in.

(16) Find the areas of the several lozenge-shaped parallelograms whose diagonals are as follows:

(1) 3·75 ft. and 4·05 ft. (1) Ans. $15\frac{3}{16}$ sq. ft.
(2) 6 ft. $2\frac{1}{2}$ in. ... 9 ft. $3\frac{1}{4}$ in. (2) Ans. 57 ft. $80\frac{1}{8}$ in.
(3) 1·05 in. ... 3·27 in. (3) Ans. 3·4335 sq. in.

(17) The diagonal of a square is 20 ft.; find the *side* approximately to 3 places of decimals. Ans. 14·142 ft.

(18) Two vessels sail from the same point, one due North at 9 knots per hour, and the other East at 11 knots; find how far they are apart in 12 hours.

Ans. 170·5512...miles.

(19) A ladder 40 feet high reaches to $\frac{2}{3}$rds of the height of a building, when placed across a street 8 yds. wide; how much must the ladder be lengthened, so that it may reach the top of the building without changing its resting-place in the street? Ans. 13·665 ft.

(20) Prove that the rectangle, whose sides are 18 units and 8 units, has a longer diagonal than the square of the same area.

(21) Shew that the area of the square, whereof the perimeter is 40 units, is greater than that of any other oblong of the same perimeter: also, shew that with the same perimeter, the greater the inequality of the sides, the less the area.

(22) In measuring a narrow rectangle, if there be any liability of error, shew that it is more important to be accurate in measuring the breadth than the length.

(23) Find how many feet of planking are required to form a single shelf, round a room 24 ft. by 16 ft., if the shelf is $1\frac{1}{2}$ ft. broad, and it is interrupted by a door and two windows, that are respectively $3\frac{1}{2}$ feet and 4 feet wide. Ans. $93\frac{3}{4}$ sq. ft.

(24) A room whose floor is 36 ft. by 27 ft. is surrounded by desks placed 1 foot from the wall; how much of the floor is enclosed within their inner boundary if they be a yard broad? Ans. 532 sq. ft.

(25) A picture, each of whose sides is 1 foot, is surrounded by a flat frame containing $1\frac{1}{4}$ sq. ft.; find the outer circumference of the frame. Ans. 6 ft.

(26) A copy-book is ruled for writing, with lines $1\frac{1}{6}$ inches apart, and with sloping transverse lines, which intersect the others at equal intervals of $\frac{3}{8}$ of an inch. Find the area of each of the equal parallelograms, into which the surface is divided. Ans. $\frac{7}{16}$ of a square inch.

(27) A hole, a yard square, is to be diminished to half its size, and still to be a square. What is the length of the side of the diminished square, and what of its diagonal? (1) Ans. 6·364 ft. nearly. (2) Ans. 1 yard.

(28) A square, and another square, one-fourth of its size, are to be formed into one square; find the proportion of its side to the side of the first square.

Ans. 1·183 : 1.

(29) A trench is cut round a camp 4 ft. deep and $5\frac{1}{2}$ feet wide; and the earth is formed into a rampart 3 ft. in perpendicular height, the face of which slopes so that its upper edge is one foot from the vertical side of the trench; find the shortest ladder that will reach

(1) from the upper edge, and (2) from the lower edge, of the trench to the top of the rampart.

(1) Ans. 7·158. (2) Ans. 9·552.

(30) Find the expense of glazing a window $5\frac{1}{4}$ ft. by 3 ft., with diamond quarries, whose diagonals are 9 and 7 inches, at 2*s*. 9*d*. per dozen.

Ans. 8*s*. 3*d*.

(31) How much paper $\frac{3}{4}$ yd. wide, will be sufficient to paper a room 22 ft. 5 in. long, 12 ft. 1 in. broad, and 11 ft. 3 in. high? And how much will it cost at $4\frac{1}{2}$*d*. per yard? (1) Ans. 115 yds. (2) Ans. £2. 3*s*. $1\frac{1}{2}$*d*.

(32) How many square feet of board will be required to make a rectangular box with lid, of which the length, breadth, and depth are $3\frac{1}{2}$ ft., $2\frac{1}{4}$ ft., and 1 ft. $2\frac{1}{4}$ in. respectively? Ans. 29 sq. ft. $58\frac{1}{2}$ sq. in.

III. OF ANGLES, CIRCULAR LINES, AND CIRCULAR AREAS.

232. As a *line* is measured by the ratio which it bears to another known *line;* and a *surface,* or *area,* by the ratio which it bears to another known *surface* or *area;* so an *angle* is measured by comparing it with another known *angle*, as the unit of measure. This unit is the *right angle.* And as the lineal *foot* is divided into parts called *inches*, which are again subdivided; and the *square foot* into *square inches*, &c., so the *right angle* is supposed to be divided into 90 equal *angles* or parts, called *degrees;* each degree into 60 equal parts called *minutes;* and each minute into 60 equal parts called *seconds.*

Hence a *right angle* is arithmetically expressed by 90 *degrees*, usually written thus, 90^{0}; half a right angle is 45^{0}; one-third of a right angle is 30^{0}; two right angles are 180^{0}; and four right angles are 360^{0}. One-fourth of a right angle is $22\frac{1}{2}^{0}$, that is, 22^{0} 30 min., usually written thus, $22^{0}\ 30'$.

And an angle which is measured by *degrees*, *minutes*, and *seconds*, is usually denoted by the marks ^{0}, $'$, $''$, placed to the right of the digits expressing the number

of such *degrees*, *minutes*, and *seconds*, respectively. Thus, 10 *degrees*, 6 *minutes*, and 21 *seconds* would be written thus, $10^{\circ}\,6'\,21''$; and would measure an angle greater than the ninth part of a right angle by $6'\,21''$.

If a certain angle cannot be exactly expressed in *degrees*, *minutes*, and *seconds*, the remainder after *seconds* is expressed in *decimal parts* of a *second*.

But although *theoretically* the *right angle* is the unit, or standard, of measure for angular magnitude, because it is an angle which meets us at every turn, and incapable of being misunderstood, yet it is plain from what has been already said, that *practically* the unit is the 90th *part* of a *right angle*, called a *degree*. Every angle we hear mentioned, or described, is said to be so many *degrees*, or *degrees* and fractions of a *degree;* the *right angle* as a *unit* disappears when we really come to work. *Practically*, therefore, we must consider the *degree*, or the 90th part of a *right angle*, as the *unit* of angular magnitude.

233. *To measure a given* angle.*

As the footrule, or some other material standard of measure, is required for measuring *lines*, and *areas*, so the unit or standard of *angular* measure being a certain angle called a *degree*, some convenient representative of this unit is required wherewith to measure a *proposed* angle. This we have in an instrument called

THE PROTRACTOR,

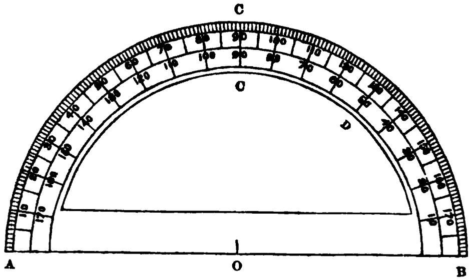
which is usually, for school purposes†, a thin plate of brass in the form of a semicircle, with a concentric segment cut out of the middle, as represented in the annexed diagram.

The semicircular band, called the *limb* of the instrument, is divided into 180 equal parts by straight lines, all of which, if produced, pass through the centre *O* of the semicircle; the outer edge of these is subdivided,

* "Given," that is, by being presented to us *traced* on a plane surface, its *arithmetical* magnitude being unknown.

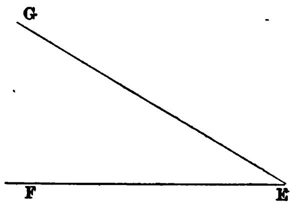
† The Protractor used by sailors and surveyors in actual work is always a *complete circle*.

each into 10 equal parts, and then the main divisions are marked 10, 20, 30, 40......180, from right to left on the *inner* rim, and the same from left to right on the outer rim—the whole limb being divided for this purpose into two rims by concentric semi-circular arcs marked on it. (The reason for the *two* graduations in a reverse order will appear afterwards). Then, if C be the point in

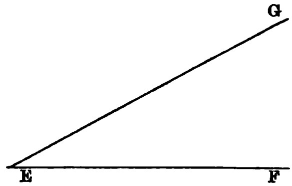


either rim marked 90, and OC be joined, it is evident that $\angle AOC = \angle BOC =$ a right angle: and if from the point O lines be drawn to each point of division in the outer edge of the limb, on the principle that in the same circle equal arcs subtend equal angles at the centre (59, Part I.), it is plain, that the right angle is divided by those lines into 90 equal parts, and that, therefore, each of these parts is a *degree* (232). Also any number of such equal parts will together make an angle which is measured by that number of *degrees*. Hence, if OD be a straight line from O meeting the *inner* circumference, for example, at the point marked 50, then $\angle AOD = 130^\circ$; and $\angle BOD = 50^\circ$.

So, then, to measure any proposed angle, as FEG, place the *Protractor* so that the centre O coincides with the vertex E, and the outer straight edge AO with FE; then you have simply to observe where the line EG meets the *outer* curved edge of the instrument; and the number of divisions of the limb from



that point to A, (which is marked upon it,) is the number of *degrees* by which the angle FEG is measured. Or, if the angle lies the other way, place the point O upon E, and OB upon EF, and mark where EG meets the outer edge of the instrument as before, in which case the figures of the *inner* rim will determine the number of divisions of the limb subtending, and therefore the number of *degrees* in, the proposed angle.



234. Conversely, we may *lay down* an angle containing any given number of *degrees*, &c.

Thus, let it be required to lay down an angle containing $37\frac{1}{2}°$; and suppose the divisions on the circumference of the *Protractor* are marked at intervals of 1°.

And 1st, Let it be required to trace the angle without regard to any proposed position. Place the *Protractor* on the paper, and draw a straight line along AB; and mark on the paper the point O. Then observe where the degrees 37 and 38 occur upon the circular arc of the *Protractor*, and mark upon the paper with a sharp point, exactly halfway between those divisions. Join the centre O and the point so taken; the required angle will then be described.

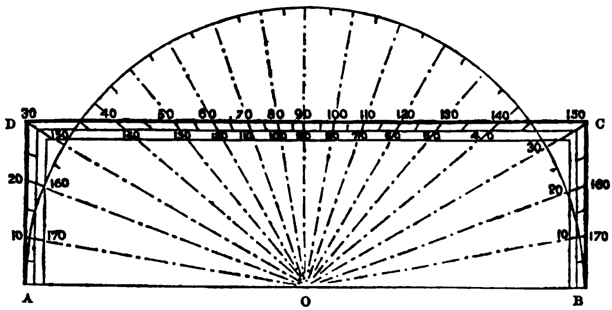
2ndly, If it be desired to draw a straight line at any proposed inclination to another line already given in position, it is only necessary to place the *Protractor* so that the given line coincides with the radius OA, or OB, and proceed as before.

235. As the intervals of *minutes* between the *degrees* cannot be marked on an ordinary instrument, we are obliged mostly to guess the number of *minutes*, whenever the bounding line of the angle falls between two consecutive *degrees;* and it is therefore advisable to diminish this source of error as much as possible.

Now, by (52, Part II.), the angle which any arc of a circle subtends at the centre of the circle is double of the angle which it subtends at the circumference; if, therefore, in measuring an angle, instead of placing the

vertex of the given angle to coincide with *O*, we make it coincide with *A*, the line which bounds the angle, instead of pointing at the true number of degrees, will point at *double* the number, and the reading of the limb must therefore be *halved*. Hence, if there be any constant error in the observation, it will be *halved*. For example, suppose the error of observation average 10′, so that an angle reckoned from *O* was put down 30° 10′, which was really 30° 20′; then, reckoned from *A* it would really be 60° 40′; but would be, with the average error, put down 60° 30′; and this, when *halved*, would be 30° 15′, giving an error of 5′, instead of 10′.

236. THE PROTRACTOR sometimes takes the form of a flat ruler, graduated after the following manner:



ABCD is a thin flat ruler in the form of a rectangle; *AB* is bisected in *O*: with centre *O* and radius *OA*, or *OB*, describe a semicircle. Divide this semicircle into 18 equal parts by lines through *O*; and mark the points of *intersection* of these lines with the *edge of the ruler*, just as the semi-circular limb in the former case was marked. When this has been done, it is obvious that the semicircle is of no further use, and may be entirely erased, leaving nothing but the ruler, which may be used precisely as before directed for measuring, or laying down, an angle.

This form of the *Protractor* is at the same time a flat ruler for drawing straight lines. A more complete form of *Protractor*, and the one mostly used in practice, when

angles are required to be measured or laid down with great accuracy, will be described among other instruments in a future chapter.

237. *To measure the* circumference *of a given circle.*

Following the method before employed in measuring a curved line (219) by means of a string or tape, the circumference of a circle which is accessible at every point may be measured. And this is the method most commonly employed, when practicable.

But it is so usual to consider a circle *given*, when its *radius* or *diameter* only is given, that it becomes necessary to measure circumferences of circles by determining what *proportion* the circumference bears to the *radius;* and as this proportion is proved to be the same for all circles, (93, Part I.), that is, a *constant* quantity, the *number* which represents it is an important number in many mathematical calculations.

It was shewn in (93) that if C, c, represent the circumferences of any two different circles, and D, d, their diameters, $C : c :: D : d$, or $C : D :: c : d$, that is, $\frac{C}{D}=\frac{c}{d}$, or the circumference of a circle bears an *invariable* ratio to its diameter, and therefore to its radius. This fixed number is commonly denoted by the Greek letter π (read *pi*), the first letter of the Greek word *periphery,* or circumference. So that if

$$\frac{C}{D}=\pi, \quad C=\pi\times D;$$

or the *circumference of a circle is equal to its diameter multiplied by* π.

But still the question remains, What is the *numerical value of* π? or, *How many times* is the diameter contained in the circumference?

Now, as this value, or number, is the same for every circle, it is obvious, that a single *accurate* measurement should be sufficient to determine it. It might appear at first sight, that nothing can be more easy than to take a perfectly constructed circle, and measure its circumference with a cord; and then measure the diameter in like manner, and find the ratio of two measurements, which will give the numerical value of π.

But, as in the case of the diagonal and side of a square before mentioned (228), so here also it is found, that the ratio of the circumference of a circle to its diameter cannot be exactly expressed in *numbers*. Each separately can be measured with perfect exactness; but *both* do not admit of being measured exactly by the *same* linear unit, however small that unit may be. This is another example of *incommensurable* lines.

Practically, $\frac{22}{7}$, or $3\frac{1}{7}$, is found to express for many purposes with sufficient exactness the value of π, the circular multiplier; $\frac{355}{113}$ is still nearer; and it will seldom be necessary to use a nearer approximate value than 3·1416.

Thus, for rough calculations,

$$\textit{circumference of circle} = \frac{22}{7} \times \textit{diameter};$$

$$\text{for finer work, } \textit{circumference} = \frac{355}{113} \times \textit{diameter},$$

$$\text{or} = 3{\cdot}1416 \times \textit{diameter};$$

using whichsoever of the two is the most convenient. (See NOTE at the end of this section, p. 255).

N.B. Although the value of π cannot be expressed without *some* error; yet that error may be made as small as we please. For the value has been calculated to many places of decimals, and is found to be, up to 20 places, as follows:

3·14159265358979323846 &c.

If then for the value of π we use $\frac{22}{7}$, that is, 3·142..., it is plain that this differs from the *true* value by a quantity less than ·01; that is, supposing the diameter to be 100 inches, the circumference, with this value of π, will be 314·2... inches; but the true value is 314·15... inches; therefore the error in this circumference from using $\frac{22}{7}$ is less than *one-tenth* of an inch.

If for π we use 3·1416, in the case before supposed, the circumference thus obtained will be 314·16 inches: whereas the more correct value is 314·1592... inches; therefore the error is less than ·001 inches, that is, less than *one-thousandth* part of an inch.

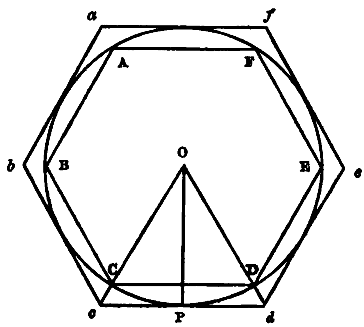
If for the value of π we use $\frac{355}{113}$, that is, 3·1415929, it is plain on comparing this with the value given before up to 20 places of decimals, that the difference is less than ·000001; and therefore, in the example before given, the error in the circumference would be less than *one-millionth* part of an inch.

And so on, by using a sufficient number of decimal places in the value of π, we can approach to the true value of the circumference, *as near as we please*, expressed in the same linear unit as the given radius or diameter.

238. *To measure the* area *of a given circle.*

We have seen (227) that the area of any regular polygon, as *ABCDEF*, is equal to the perimeter × half the perpendicular from the centre of the circumscribing circle upon one of the sides. So also, if the polygon had been described *about* the circle, as *abcdef*, the area would be equal to the perimeter of the polygon × half the perpendicular upon one of its sides, (which is *OP*, the radius of the circle).

Now the area of the circle is evidently greater than



the former polygon, and less than the latter; but if the number of the sides be indefinitely increased in each case, and therefore the length of each indefinitely diminished, the perimeter of each polygon approaches to the circumference of the circle, and the area of the circle is the *ultimate value* of the area of either polygon, when the number of the sides is indefinitely great.

Hence, putting the circumference of the circle for the perimeter of the polygons, and the radius for the perpendicular, we have

$$\textit{area of circle} = \textit{circumference} \times \tfrac{1}{2}\,\textit{radius},$$

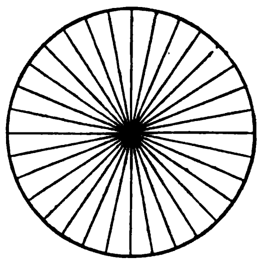
$$= \pi \times \textit{twice radius} \times \tfrac{1}{2}\,\textit{radius},\ (237),$$

$$= \pi \times (\textit{radius})^2.$$

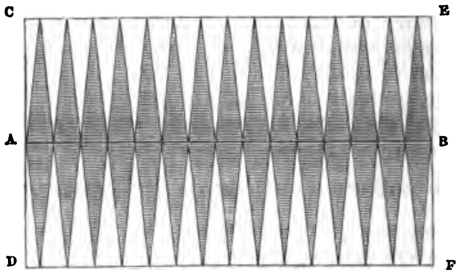
COR. Since π does not admit of being found in a terminating, or a recurring, decimal (237), therefore, the area of a circle does not admit of being converted into a rectangle without error; and therefore, cannot be *exactly* measured by square units. Hence arose the impossible problem of *squaring the circle*, as it is called, which means finding a *square* numerically equal to a proposed circle.

239. That the area of a circle is equal to *half circumf.* $\times$ *rad.* may be thus exhibited *to the eye.*

Divide each half of the circle into any the same number of equal sectors, and draw the chord at the base of each sector, forming as many isosceles triangles as there are sectors. Take a straight line *AB*, and place these triangles on it in juxtaposition, having their bases coinciding with *AB*, one-half above and the other half below *AB*, as shewn by the dark triangles in the diagram. Then it is plain, that as the *number* of sectors is increased, *AB* approaches nearer and nearer to the semi-circumference of the circle, and the altitude of each triangle to the radius of the circle.



Through A and B draw CAD and EBF at right angles to AB, and through the vertices of the triangles CE and DF parallel to AB. Then it is easily seen, that



the sum of the dark triangles is *half* the rectangle $CDEF$, whatever the number of them may be. But when that number is increased indefinitely, the aggregate area of the triangles is the area of the circle, whilst AB is the semi-circumference, and AC the radius;

$$\therefore \text{ area of circle} = \tfrac{1}{2} AB \times CD,$$
$$= AB \times AC = \tfrac{1}{2} \text{ circumf.} \times \text{rad.}$$

240. *To measure a given circular* arc.

1st. Suppose the *centre* of the circle given; and draw the radii from it to both extremities of the arc. Then, with the *Protractor,* or by some other means, measure the *angle* contained by these radii, and let it be expressed by A^0; measure also the radius, if it be not already known. Then, since (84), in the same circle, any two arcs are proportional to the angles which they subtend at the centre, the proposed arc : whole circumference :: A° : 360°,

$$\text{or, arc} : 2\pi \times \text{rad.} :: A^\circ : 360^\circ;$$

$$\therefore \text{ arc} = 2\pi \times \text{rad.} \times \frac{A}{360}.$$

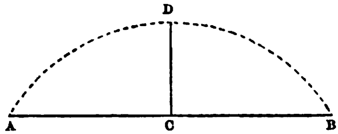
Ex. Let the arc subtend at the centre an angle of 36°, and let the radius be 7 feet, then

$$\text{the arc} = 2\pi \times 7 \times \frac{36}{360} = 2 \times \frac{22}{7} \times 7 \times \frac{1}{10} \text{ feet,}$$
$$= \frac{44}{10} = 4\tfrac{2}{5} \text{ feet.}$$

2nd. Suppose the given arc simply traced on a plane surface, but the centre of the circle not known. The centre may readily be found by (147, Part II.); then proceed as before.

3rd. Suppose the centre of the circle to be inaccessible, as in the case of a vertical section of a railway-bridge.

Measure the chord of the whole arc, and the perpendicular distance of the highest point of the arc from that chord: lay down on paper these two lines in proper proportion according to these measurements, as AB, and CD in the annexed diagram; that is, if the chord of the given arc be 30 feet, suppose, and the greatest height of the arc be 10 feet, make AB equal to $1\frac{1}{2}$ in., and



CD (drawn from the middle point between A and B, at right angles to AB), equal to half an inch. Then, by (134, Part II.) construct the circle whose circumference shall pass through the three points A, B, D (there is only *one* such circle, see Cor. 1, p. 118); find the centre of this circle (50, Part I.); and proceed as in the 1st case, remembering that in the result 1 inch represents 20 feet.

COR. It has been shewn, that the length of any circular arc which subtends an angle of $A°$ at the centre is equal to $2\pi \times \text{rad.} \times \frac{A}{360}$; therefore the length of the arc which subtends an angle of 1°

$$= 2 \times \frac{22}{7} \times \frac{1}{360} \times \text{rad.},$$

$$= \frac{11}{630} \times \text{rad.} = \cdot 01746 \times \text{rad. } \textit{nearly}.$$

Hence, since in the same circle, or in equal circles, arcs are proportional to the angles which they subtend at the centre, an arc, which subtends an angle of $A°$, will be equal to $A \times$ arc of 1°,

$$= A \times \cdot 01746 \times \text{rad. } \textit{nearly}.$$

Ex. Let the arc subtend at the centre an angle of 36^0, and let the radius be 7 feet, then

the arc $=36\times7\times\cdot01746=4\cdot3999$ ft. $=4\cdot\dot{4}$ ft. *nearly.*

The Rule in this case is—*Multiply the number of degrees which the arc subtends at the centre by the radius, and the product by* ·01746; *the result will be the length of the arc expressed in the same unit as the radius.*

N.B. If the arc subtends an angle of degrees *and* minutes, or degrees, minutes, *and* seconds, the whole must be converted into a *decimal*, with a *degree* for the unit, before the Rule is applied.

Ex. Suppose an arc subtends an angle at the centre of $10^0 36'$, and the radius is a mile; then, since

$$10^0 36' = 10\tfrac{36}{60}^0 = 10\cdot6^0,$$

$$\therefore \text{ length of arc} = 10\cdot6\times1\times\cdot01746 \text{ miles},$$

$$= \cdot185076 \text{ miles},$$

$$= 325\tfrac{3}{4} \text{ yards } \textit{nearly}.$$

241. *To measure a given* sector *of a circle.*

Let the arc of the given sector subtend an angle of A^0 at the centre; and *suppose* the whole circle divided into 360 *equal* sectors; then each of these sectors will have an arc subtending an angle of 1^0 at the centre; and it is plain, that

$$\text{given sector} : \text{whole circle} :: A^0 : 360^0;$$

$$\therefore \text{ given sector} = \pi\times(\text{rad.})^2\times\frac{A}{360}, \ (238),$$

$$= 2\pi\times\text{rad.}\times\frac{A}{360}\times\frac{\text{rad.}}{2},$$

$$\text{or area of sector} = \frac{\textit{length of arc}\times\textit{rad.}}{2}, \ (240).$$

Hence, if the number of *degrees* which the arc subtends at the centre, and the radius, be given, we use the formula

$$\text{area of sector} = \pi\times(\text{rad.})^2\times\frac{A}{360}.$$

But if the *length of the arc,* and radius, be given, we use the formula

$$\text{area of sector} = \frac{\textit{length of arc} \times \textit{rad.}}{2}.$$

Ex. 1. The arc of a sector subtends at the centre an angle of 60°, and the radius is 10 feet; required the area of the sector.

$$\text{Area of sector} = \frac{22}{7} \times 100 \times \frac{60}{360},$$

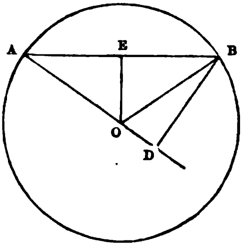
$$= \frac{11 \times 100 \times 1}{21} = \frac{1100}{21} = 52{\cdot}38 \text{ feet } \textit{nearly}.$$

Ex. 2. The length of the *arc* of a sector is 16 yards, and the radius $12\frac{1}{2}$ yards; required the area of the sector.

$$\text{Area of sector} = \frac{16 \times 12{\cdot}5}{2} = 8 \times 12{\cdot}5 = 100 \text{ sq. yds.}$$

242. *To measure a given* segment *of a circle.*

Let ABC be the given segment; and if the centre O of the circle be not given, let it be found by (147, Part II.). Join AO, BO; then it is clear, that the area of the segment ABC is equal to the sector $AOBC$ diminished by the triangle AOB; that is, if the given segment be less than a semicircle. And if the segment be greater than a semicircle, the only difference will be that the triangle must be *added* to the sector instead of being subtracted.



Hence the sector, $AOBC$, which is bounded by the same arc as the given segment, is to be measured by (241); and the triangle ABO by one of the usual methods; from which two results the area of the segment is readily determined.

Thus, segment ABC = sector $AOBC$ − triangle AOB,

$$= \frac{AO \times \text{arc } ACB}{2} - \frac{AO \times BD}{2},$$

(*BD* being a perpendicular from *B* on *AO*, or *AO* produced)

$$=\frac{AO}{2}\times(\text{arc } ACB-BD).$$

Or, if *OE* be drawn perpendicular to *AB*,

$$\text{segment } ABC=\frac{AO\times\text{arc } ACB}{2}-\frac{AB\times OE}{2}.$$

QUESTIONS AND EXERCISES. H.

(1) What is the unit generally adopted in the measurement of *angles*, 1st, theoretically, and 2ndly, practically?

(2) Give the subdivisions of the smaller of these two units.

(3) Describe the instrument used for measuring proposed angles, and laying down on paper angles whose numerical values are known.

(4) Mention the angles which can be most readily laid down without the use of any instrument specially constructed for that purpose.

(5) State the relation which exists between the circumference, and the radius, of every circle.

(6) What is the ratio between the circumference of a circle, and its *diameter?* Is the quantity expressing that ratio a line, or an abstract number?

(7) Describe the mode whereby we approximate to the *area* of a circle.

(8) What is the relation which subsists betwixt the area of a circle and its radius? Is that relation such, that if the radius be doubled, the area will be doubled? If not, *how much* is the area increased?

(9) Is the fraction which expresses the quotient of the area of a circle by its radius, a line, or an abstract number?

(10) Write down the fraction which expresses the quotient of the area of a circle by its circumference.

(11) State the difficulty we experience in finding the *exact* value of any circular area or circumference. Do we meet with any similar difficulty in the measurement of straight lines?

(12) When using the instrument called a *Protractor*, shew how any *error* of observation, made in determining an angle in the usual way, may be diminished by one half.

(13) Compare the effects of using the two numbers commonly employed for π, viz. $\frac{22}{7}$, and 3·14159, in finding the area of a circle of $7\frac{1}{2}$ inches radius.

Ans. The difference is ·07127 sq. in.

[In the following examples the value of π has been taken as $\frac{22}{7}$.]

(14) If the circumference of a circle be 4 poles, what is the radius?

Ans. $3\frac{1}{2}$ yds.

(15) What must be the radius of a circle, so that the circumference shall be $1\frac{1}{2}$ poles?

Ans. 1 yd. $11\frac{1}{4}$ in.

(16) When the area of a circle is a square perch, what is its radius?

Ans. 3·1024... yds.

(17) The diameter of a circle is $3\frac{1}{2}$ yds.; what is the circumference of the circle, and what its area?

(1) Ans. 11 yds. (2) Ans. $9\frac{5}{8}$ sq. yds.

(18) Find the measure of the radius of the circle, of which the quadrant contains $28\frac{2}{7}$ sq. yds.

Ans. 6 yds.

(19) Find the circumference, approximately to 3 places of decimals, of the circle, of which the area is 16 sq. yds.

Ans. 14·182 yds.

(20) Given that the side of an equilateral hexagon inscribed in a circle is equal to the radius, find by what portion of the radius the semi-circumference of the circle exceeds the sum of three sides of the hexagon.

Ans. $\frac{1}{7}$.

(21) Find the length of the minute-hand of a clock, the extremity of which moves over an arc of 10 inches in $3\frac{3}{4}$ minutes.

Ans. $25\frac{5}{11}$ in.

(22) The sum of the interior angles of a regular polygon is 1800°; find how many sides it has, (See 86, Part. I.).

Ans. 12.

(23) The area of a circle is 29 sq. ft.; what is the area of another circle whose diameter is three-sevenths of the diameter of the former?

Ans. 5·3265 sq. ft.

(24) Find the area of a segment of a circle, whose arc subtends an angle of 60° at the centre, the diameter of the circle being 10 feet.

Ans. 2·27 sq. ft.

(25) Prove that a cord, with its ends joined, will inclose a greater area when in the form of a circle than in the form of a square.

PROBLEMS.

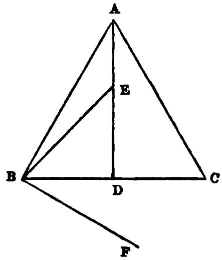
PROB. 1. To find the numerical value of each of the angles of a right-angled isosceles triangle.

By the supposition, one of the three angles is 90°, and the other two are equal to one another; but all the angles of any triangle are together equal to two right angles, therefore the two equal angles, in this case, are together equal to *one* right angle, that is, each of them is half a right angle, or 45°.

Hence, conversely, if we have a right-angled triangle, in which one of the acute angles is known to be 45°, we may conclude, that the other angle is also 45°; and further that the sides containing the right angle are *equal*. This property is very useful in enabling us to find the heights of lofty buildings, or the distance between two inaccessible points, as will appear hereafter.

PROB. 2. To construct angles of 60°, 30°, 15°, and 75°, without the aid of the *Protractor*.

Describe an equilateral triangle ABC, by (23, Part I.); then we know that this triangle is also equiangular; and since the three angles are together equal to two right angles, or 180°, (37, Part I.) therefore each of them is 60°.



Bisect BC in D; and join AD; then $\angle BAC$ will be bisected by AD; and therefore $\angle BAD=30°$.

Again, from DA cut off DE equal to DB, and join BE; then $\angle DEB=\angle DBE$; and, by Prob. 1, each $=45°$; therefore

$$\angle ABE=\angle ABD-\angle DBE=60°-45°=15°.$$

Again, draw BF at right angles to AB; then

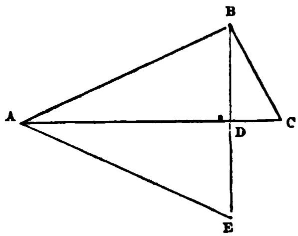
$$\angle EBF=\angle ABF-\angle ABE=90°-15°=75°.$$

Thus

$$\angle ABC=60°, \ \angle BAD=30°, \ \angle ABE=15°, \text{ and } \angle EBF=75°.$$

PROB. 3. To find the area of a triangle, two sides of which are given, and contain an angle of 30°.

Let ABC be the triangle, in which AB and AC are known, and $\angle BAC=30°$.



Draw BD perpendicular to AC; then, since AC is known, and the area of the triangle

$$=\frac{AC\times BD}{2}, \ (225),$$

therefore, when BD is found, the area is known.

Produce BD to E, making DE equal to DB, and join AE; then

$$\angle DAE=\angle DAB=30°; \text{ and } \angle EAB=60°.$$

Also, since

$$\angle ADB=90°, \text{ and } \angle BAD=30°; \therefore \angle ABD=60°.$$

Hence the triangle ABE is equilateral, and

$$BD=\tfrac{1}{2}BE=\tfrac{1}{2}AB.$$

Therefore the area of ABC

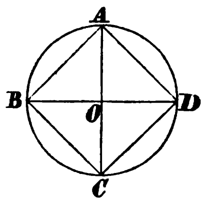
$$=\frac{AC\times BD}{2}=\frac{AC\times AB}{4};$$

that is, the area of a triangle, which has one angle of 30°, is equal to one-fourth of the product of the two sides containing that angle.

Ex. Let $AC=24$ yds., and $AB=17{\cdot}6$ yds.; then

$$\text{area of } ABC=\frac{24\times17{\cdot}6}{4}=105{\cdot}6 \text{ sq. yds.}$$

PROB. 4. To find the length of a side of the square '*inscribed*' in a given circle.



Let $ABCD$ be the given circle, and O its centre: AC and BD two diameters at right angles to each other. Join AB, BC, CD, DA; then we have a square inscribed (155, Part II.). Now,

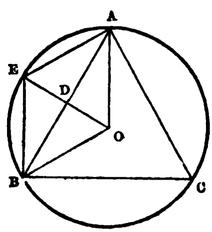
$$AB^2=AO^2+BO^2=2AO^2=2\times(\text{rad.})^2;$$

$$\therefore\ AB=\text{rad.}\times\sqrt{2};$$

or *the side of the inscribed square is equal to the radius multiplied by* $\sqrt{2}$.

If rad.=1, the side of the square $=\sqrt{2}$.

PROB. 5. To find the length of a side of the equilateral triangle '*inscribed*' in a given circle.



Let O be the centre of the given circle, and ABC an equilateral triangle inscribed in it. From O draw OD perpendicular to AB, and produce it to meet the circumference in E. Join AE, BE. Then it is easily shewn that $AEBO$ is a lozenge, and that

$$OD=\tfrac{1}{2}OE=\tfrac{1}{2}\text{ rad.}$$

But $AB=2AD=2\sqrt{OA^2-OD^2}$,

$$=2\sqrt{OA^2-\frac{1}{4}OA^2}=2\sqrt{\frac{3}{4}OA^2};$$

$$\therefore\ AB=OA\sqrt{3},$$

$$=\text{rad.}\times\sqrt{3}.$$

If rad.=1, the side of the inscribed equilateral triangle $=\sqrt{3}$.

Prob. 6. To shew that the side of a square, together with the side of an equilateral triangle, both inscribed in the same circle, is equal to half the circumference of the circle, *nearly*.

By Prob. 4, the side of square

$$=\text{rad.}\times\sqrt{2}=\text{rad.}\times 1\cdot 414\ldots\ldots$$

By Prob. 5, the side of triangle

$$=\text{rad.}\times\sqrt{3}=\text{rad.}\times 1\cdot 732\ldots\ldots;$$

$$\therefore\ \text{sum of the two}=\text{rad.}\times(1\cdot 414+1\cdot 732),$$

$$=\text{rad.}\times 3\cdot 146,\ \text{nearly}.$$

But half the circumference of the circle

$$=\text{rad.}\times 3\cdot 14159;$$

therefore the side of the square added to the side of the triangle does not differ from the semi-circumference of the circle by a quantity so great as $\cdot 005$, or $\frac{5}{1000}$, that is, the 200th part of ~~a unit~~ the radius.

Prob. 7. To find the numerical value of the angle at the centre of a circle subtended by an arc equal to the radius.

By (240) we know, that the length of an arc subtending an angle of A° at the centre

$$=2\pi\times\text{rad.}\times\frac{A}{360};$$

therefore, in this case,

$$\text{rad.}=2\pi\times\text{rad.}\times\frac{A}{360},\ \text{to find } A.$$

But this equality cannot hold, unless

$$2\pi \times \frac{A}{360} = 1;$$

$$\therefore A = \frac{360}{2\pi} = \frac{360}{2 \times 3\cdot1416} = \frac{180^\circ}{3\cdot1416},$$

$$= 57^\circ\, 17'\, 45'', \text{ nearly.}$$

Prob. 8. To construct a rectangle, or triangle, which shall be equal to a given circle.

Assuming that the area of the given circle is

$$\frac{22}{7} \times (\text{rad.})^2, \ (238),$$

divide the given radius into 7 equal parts (168, Part II.); then construct a rectangle, of which the base is 22 of such parts, and the height 7, that is, the radius. The area of this rectangle

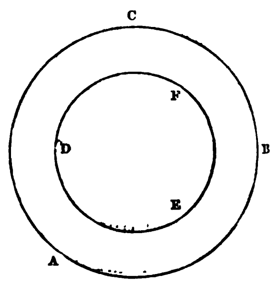
$$= \text{base} \times \text{height } (223),$$

$$= 22 \times \frac{\text{rad.}}{7} \times \text{rad.},$$

$$= \frac{22}{7} \times (\text{rad.})^2 = \text{area of given circle.}$$

For a *triangle* of equal area, make the base the same, and the height equal to double the radius, that is, to the diameter.

Prob. 9. To measure the area of a circular *ring*.



Let it be required to find the area of the ring enclosed by the two concentric circles *ABC*, *DEF*. It is plain, that this area will be found by subtracting the area of the smaller circle from that of the greater. So that, if R represents the radius of the greater, and r the radius of the smaller, circle, we have, by (238),

$$\text{area of greater circle} = \pi \times R^2,$$

$$\ldots\ldots \text{ smaller } \ldots\ldots = \pi \times r^2;$$

$$\therefore \text{ area of ring} = \pi \times R^2 - \pi \times r^2 = \pi \times (R^2 - r^2);$$

that is, *the square of the smaller radius must be subtracted from the square of the greater, and the difference multiplied by the number* π, *as given in* (237).

Ex. Let the inner and outer radii of a ring be 30 feet, and 35 feet, respectively; then

$$R^2 = 1225, \text{ and } r^2 = 900;$$

and the difference $= 325$; therefore area of ring

$$= 3{\cdot}1416 \times 325 \text{ sq. feet},$$

$$= 1021{\cdot}02 \text{ sq. feet}.$$

Note. By the well-known rule, which can be proved both geometrically and algebraically, viz. that *the difference of the squares of any two numbers is equal to the product of their sum and difference,* the trouble of squaring *large* numbers may, in this instance and in some others, be avoided. Thus $R^2 - r^2$ is equal to $R + r$ multiplied by $R - r$, whatever numbers R and r stand for. And, taking the above Ex., $R + r = 65$, $R - r = 5$, therefore $R^2 - r^2 = 5 \times 65 = 325$, as before.

But the advantage of this device will be better seen in such an example as the following:—

Ex. The outer and inner radii of a circular ring are 365 yards, and 355 yards, respectively; find the area.

Here $R + r = 720$, and $R - r = 10$;

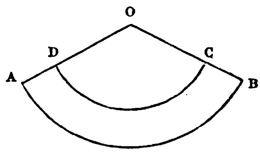
$$\therefore R^2 - r^2 = 720 \times 10 = 7200,$$

$$\text{and area of ring} = \pi \times 7200 \text{ sq. yards}.$$

It is further to be observed, that, although in the above problem the circles were said to be *concentric,* this is not a necessary condition. The same results precisely will be obtained, provided one circle be wholly *within* the other.

PROB. 10. To measure the area of a *portion* of a circular ring, as $ABCD$ in the diagram below.

It is plain, that the area of $ABCD$ is equal to the difference of the *sectors* OAB, and OCD, O being the common centre of the two arcs AB and CD.



If, then, the radii are represented by R, r; we have, by (241),

$$\text{sector } OAB = \tfrac{1}{2} AB \times R,$$

$$\text{and } \ldots\ldots\ldots \; OCD = \tfrac{1}{2} CD \times r;$$

$$\therefore \text{ area } ABCD = \tfrac{1}{2} AB \times R - \tfrac{1}{2} CD \times r.$$

If, however, the radii are not given and cannot easily be found, this result must be modified as follows :—(The process will be readily understood by those who have a little knowledge of Algebra :)

By (240), $AB : CD :: R : r$;

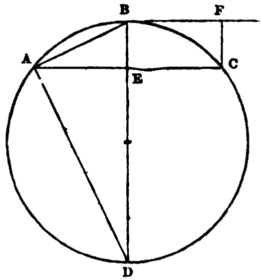
$$\therefore AB \times r = CD \times R, \text{ (74, Part I.)},$$

$$\text{or } \tfrac{1}{2} AB \times r = \tfrac{1}{2} CD \times R;$$

$$\begin{aligned} \text{and area } ABCD &= \tfrac{1}{2} AB \times R - \tfrac{1}{2} CD \times r, \\ &= \tfrac{1}{2} AB \times R - \tfrac{1}{2} AB \times r + \tfrac{1}{2} AB \times r - \tfrac{1}{2} CD \times r, \\ &= \tfrac{1}{2} AB \times R - \tfrac{1}{2} AB \times r + \tfrac{1}{2} CD \times R - \tfrac{1}{2} CD \times r, \\ &= \tfrac{1}{2} AB \times (R - r) + \tfrac{1}{2} CD \times (R - r), \\ &= \tfrac{1}{2} (AB + CD) \times (R - r), \\ &= \tfrac{1}{2} (AB + CD) \times AD; \end{aligned}$$

that is, *half the sum of the two arcs multiplied by the distance between them,* as in an ordinary trapezium. And, since this is true for *any* portion of a ring, bounded as above, it follows, that it is true also for the *complete* ring, viz. the area of the ring is equal to half the sum of the circumferences multiplied by the distance between them.

PROB. 11. To find the radius of a segment-arch, having given the span and the rise.



Let ABC be the arch, AC the span, BE, which bisects AC at right angles, the rise. Complete the circle $ABCD$ (147, Part II.); produce BE to meet the circumference in D; join AB and AD. Then BD is a diameter (49, Part I.), and $\angle BAD$ is a right angle (54, Part I.).

Therefore

$$BD : AB :: AB : BE \text{ (72, Part I.)};$$

$$\therefore BD \times BE = AB^2 \text{ (74, Part I.)};$$

$$\therefore BD = AB^2 \div BE.$$

$$\text{But } AB^2 = BE^2 + AE^2;$$

$$\therefore AB^2 \div BE = BE + \frac{AE^2}{BE};$$

$$\therefore BD = BE + \frac{AE^2}{BE};$$

that is, *to find the diameter, add the rise to the quotient of the square of half the span divided by the rise.*

COR. Hence may be obtained the diameter of any circular area, as a fish-pond, which cannot be traversed.

For, using the above diagram, from B draw BF parallel to AC, and from C draw CF parallel to BE; then since

$$BF = EC = AE, \text{ and } CF = BE,$$

we have, by the last case,

$$\text{diameter} = CF + \frac{BF^2}{CF}.$$

Ex. Suppose

$BF = 9{\cdot}6$ ft., and $CF = 3{\cdot}5$ ft., then

$$BD = 3{\cdot}5 + \frac{(9{\cdot}6)^2}{3{\cdot}5} = 3{\cdot}5 + 26{\cdot}331 = 29{\cdot}831 \text{ ft.}$$

If the *circumference* of the pond be known, we can, of course, readily obtain the diameter;

$$\text{for circumf.} = \pi \times \text{diameter};$$

$$\therefore \text{ diameter} = \text{circumf.} \div \pi.$$

PROB. 12. Having given a portion of a board cut by a circular saw, to find the diameter of the saw.

The teeth of the saw will leave very distinct circular arcs on the face of the board; therefore, taking a portion of any one of them, as ABC, measure AC, and also BE (see Prob. 11); then the diameter of the saw will be the diameter of the circle of which ABC is an arc, that is,

$$\text{diameter of saw} = BE + \frac{AE^2}{BE}.$$

Ex. Suppose $AC = 12$ in., and $BE = 1\frac{7}{8}$ in.; then

$$\begin{aligned}\text{diameter of saw} &= 1{\cdot}875 + \frac{36}{1\frac{7}{8}},\\ &= 1{\cdot}875 + \frac{96}{5},\\ &= 1{\cdot}875 + 19{\cdot}2,\\ &= 21{\cdot}075 \text{ in.}\end{aligned}$$

PROB. 13. To find the number of degrees in the angle contained by two adjacent sides of a given regular polygon.

By (86, Part I.), all the angles together of a polygon are equal to twice as many right angles as the polygon has sides, diminished by 4 right angles. Therefore

all the angles of a pentagon $= 6 \times 90^\circ$,

.... hexagon $= 8 \times 90^\circ$,

.... octagon $= 12 \times 90^\circ$;

and so on;

$$\therefore \textit{each} \text{ angle of } \textit{regular} \text{ pentagon} = \frac{6\times 90^\circ}{5} = 108^\circ,$$

$$\text{.......................... hexagon} = \frac{8\times 90^\circ}{6} = 120^\circ,$$

$$\text{.......................... octagon} = \frac{12\times 90}{8} = 135^\circ;$$

and so on.

Conversely, if the angle contained by two adjacent sides of a regular polygon be given, the number of *sides* may be found. For, if A° be the given angle, and each side of the polygon be produced (see 86, Cor. 2, Part I.), the *exterior* angle in each case will be $180^\circ - A^\circ$, and the number of such angles will be the same as the number of sides. But the *sum* of the exterior angles is equal to 4 right angles $= 360^\circ$.

$$\therefore \text{number of sides} = \frac{360}{180 - A}.$$

Thus, if each of the angles of a regular polygon be

$$108^\circ, \text{ number of sides} = \frac{360}{180-108} = \frac{360}{72} = 5;$$

$$120^\circ, \text{ ....} = \frac{360}{180-120} = \frac{360}{60} = 6;$$

$$135^\circ, \text{ ....} = \frac{360}{180-135} = \frac{360}{45} = 8;$$

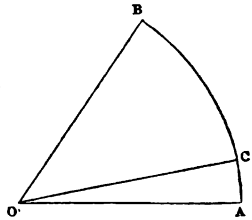
and so on.

PROB. 14. To express the *ratio* between two given arcs of the same circle, or of equal circles.

If the *centre* of the circle be not given, it must be found, by (50, Part I.); then draw the radii belonging to the extremities of the given arcs; and, by means of a *Protractor* or otherwise, measure the *angles* contained by those radii for each arc. Reduce the numerical values of *both* angles to the lowest name mentioned in *either* of them (that is, if either of them contain *seconds*, *both* must be reduced to *seconds*, or the same multiple of a *second*). Then, since in the same circle, or in equal circles, any two arcs are proportional to the *angles* which they sub-

tend at the centre, the ratio required will be the same as that of the *angles* above-named.

Thus, to find the ratio of the arc AB to the arc AC, in the same circle, subtending at the centre the angles AOB, AOC, respectively. Let the angles AOB, AOC be measured, and suppose them to be $45°35'$, and $7°7'6''$. These, when reduced to portions of $6''$ (that is, making $6''$ the unit), become respectively 27350 and 4271; therefore



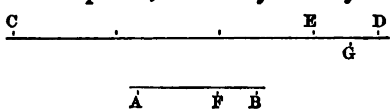
$$\text{arc } AB : \text{arc } AC = \frac{27350}{4271}.$$

A *rough approximation* would be found from $\frac{27300}{4200}$, that is, $\frac{273}{42}$, or $\frac{91}{14}$, or $\frac{13}{2}$. Whether such an approximation would answer the purpose, must depend upon the accuracy required in the particular work in hand.

A very useful mode of computing the magnitudes of straight lines, angles, circular arcs, &c. may be fitly introduced here, although it has not hitherto been noticed by writers on *Mensuration*. It consists in comparing the proposed magnitude with some unit of the same kind, by means of *Continued Fractions*. Thus—

PROB. 15. Let it be required to measure the straight line CD in terms of the unit AB; that is, to find the numerical value of the fraction $\frac{CD}{AB}$ approximately, to any required degree of accuracy.

Open the compasses, until they exactly embrace AB.



Then, with the compasses thus fixed, step along *CD*, marking the intervals, each equal to *AB*; and suppose there are *three* such intervals in *CD*, with a remainder *ED*, less than *AB*. Then

$$CD = 3AB + ED,$$

$$\text{or } \frac{CD}{AB} = 3 + \frac{ED}{AB}.$$

Now set the compasses to *ED*, and step along *AB*; at intervals each equal to *ED*, and suppose *ED* to be contained twice in *AB* with remainder *FB*. Then

$$AB = 2ED + FB;$$

$$\therefore \frac{AB}{ED} = 2 + \frac{FB}{ED} \ldots\ldots (1).$$

Similarly, let *ED* be divided by *FB*, and let *FB* go once with a remainder *GD*, so that

$$\frac{ED}{FB} = 1 + \frac{GD}{FB} \ldots\ldots (2).$$

Let this process be continued, till there either be no remainder, or the remainder be so small that it may be neglected. Suppose *GD* to be the last remainder; so that *GD* goes twice exactly in *FB*, or $\frac{FB}{GD} = 2$.

We have, then,

$$\frac{CD}{AB} = 3 + \frac{ED}{AB} = 3 + \cfrac{1}{\cfrac{AB}{ED}},$$

$$= 3 + \cfrac{1}{2 + \cfrac{FB}{ED}}, \text{ by (1)},$$

$$= 3 + \cfrac{1}{2 + \cfrac{1}{\cfrac{ED}{FB}}},$$

$$=3+\cfrac{1}{2+\cfrac{1}{1+\cfrac{GD}{FB}}}, \text{ by (2)},$$

$$=3+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2}}}.$$

This 'continued fraction' is reduced to an ordinary fraction by commencing at the lower extremity thus:—

$$1+\frac{1}{2}=\frac{3}{2}, \quad \cfrac{1}{1+\cfrac{1}{2}}=\frac{2}{3}, \quad 2+\frac{2}{3}=\frac{8}{3};$$

$$\therefore \frac{CD}{AB}=3+\frac{3}{8}=3\tfrac{3}{8}.$$

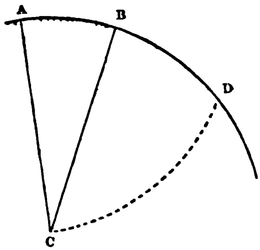
But the great use of this method is not in enabling us to obtain the *exact* measure of the ratio $CD : AB$, but an *approximate* value, which shall be *as near as we please* to the true value. Thus, in this case, 3 is an approximate value; $3+\frac{1}{2}$, or $3\frac{1}{2}$, is a nearer value; and $3+\cfrac{1}{2+\cfrac{1}{1}}$, or $3\frac{1}{3}$, is nearer still. And similarly, whatever be the length of the continued fraction, by stopping at any particular quotient, and neglecting the remainder, an *approximate* value of the fraction is obtained, which differs less and less from the true value according as more of the continued fraction is taken into account. And it will be seen in the above case, (as it is indeed in all others,) that the approximate values 3, $3\frac{1}{2}$, $3\frac{1}{3}$, taken in order, are alternately less and greater than the true value $3\frac{3}{8}$. The last approximate value, $3\frac{1}{3}$, differs from the true value, $3\frac{3}{8}$, by only one twenty-fourth of the unit.

PROB. 16. By a similar method to that employed in the last Prob., to compare two given *angles* with each

other, or to find the measure of a proposed *angle* in terms of some given unit.

Any one of the following ways may be employed to measure the angle ACB:—

(1) With centre C, and greatest radius that can be conveniently used, describe an arc of a circle not less than the sixth part of the whole circumference, and cutting the two lines which bound the given angle in A and B. Then with centre A, and the same radius as before, describe another arc intersecting the former in D; AD* is an arc of 60°.



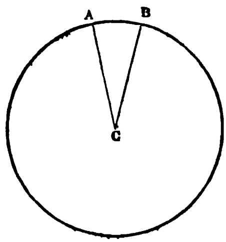
Then, since in the same circle arcs are proportional to their chords, by stepping AB, with the compasses, along AD, as described in Prob. 15 for straight lines, the *arc* AB may be compared with the *arc* AD, just as the straight line AB, in the former case, was compared with the straight line CD.

$$\text{And } \frac{\angle ACB}{\angle ACD} = \frac{\text{arc } AB}{\text{arc } AD};$$

$$\therefore \angle ACB = \frac{\text{arc } AB}{\text{arc } AD} \times 60^\circ.$$

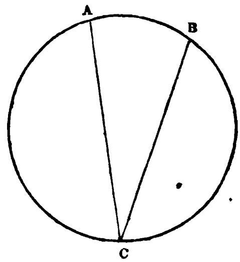
(2) Or, compare the arc AB with the *whole circumference*, by stepping the chord AB all round; then

$$\angle ACB = \frac{\text{arc } AB}{\text{whole circumf.}} \times 360^\circ.$$



This, of course, implies a large surface to work upon.

* It is usual to speak of an *arc* of so many *degrees*, meaning an arc which subtends that *angle* at the *centre*.



(3) Or, with any centre, describe as large a circle as the case admits of, with the circumference passing through C, the vertex of the given angle, and intersecting the straight lines, which bound the angle in A and B. Then compare the arc AB with the whole circumference, as before, and (52, Part I.),

$$\angle ACB = \frac{\text{arc } AB}{\text{whole circumf.}} \times 180^\circ.$$

EXERCISES I.

[In these Exercises π is taken to be 3·1416, when it is not otherwise stated.]

(1) Find the area of an isosceles triangle, in which the angle contained by the equal sides is 120°, and the altitude of the triangle is 6 ft. 3 in.

Ans. 67·656 sq. ft.

(2) Two adjacent sides of a triangle measure 35·6, and 44·2, yards, and contain an angle of 30°; find the area of the triangle. Ans. 393·38 sq. yds.

(3) Having a given circle traced out before you, shew how to trace another, whose *circumference* shall be exactly 4 times that of the former. How would you trace one whose *area* shall be 4 times that of the first?

(4) The circumference of a circle is 38 inches; find the length of a side of the greatest equilateral triangle which can be cut out of it.

Ans. 10·475 in.

(5) The radii of two circles are 5, and 12; find the radius of another circle, whose area shall be exactly equal to the sum of the areas of the other two. Ans. 13.

(6) In cutting out the greatest square from a given circular board, how much of the material is wasted? $\left(\pi = \frac{22}{7}\right)$.

Ans. $\frac{4}{11}$, or a little more than one-third, of the whole.

(7) In cutting out the greatest equilateral triangle from a given circular board, how much of the material is wasted? $\left(\pi = \frac{22}{7}\right)$.

Ans. ·586, or more than half, of the whole.

(8) The diagonal of a square is 45 yards; find the area of the inscribed circle. Ans. 795·2175 sq. yds.

(9) Compare the area of a square with the sum of the semi-circles described upon its sides.

Ans. $2 : \pi$.

(10) Two radii of a circle are at right angles to each other, and a chord is drawn joining their extreme points; compare the segments into which the circle is thus divided. $\left(\pi = \frac{22}{7}\right)$. Ans. 10 : 1.

(11) If the area of a circle be 16 sq. yds., find the area of a sector of the circle, whose arc subtends at the centre an angle of 75°. Ans. $3\frac{1}{3}$ sq. yds.

(12) The radius of a circle is 25 feet, and the angle of a *sector* of it contains 63°; find the length of the arc, and thence the area of the sector.

(1) Ans. $27\frac{1}{2}$ ft. nearly. (2) Ans. $343\frac{3}{4}$ sq. ft.

(13) Find the length of an arc of 17° 10′, the diameter of the circle being 6 feet.

Ans. ·8988...... ft.

(14) A portion of wood cut by a circular saw shews an arc of a circle on its face made by the teeth of the saw, of which the chord is 9 inches; and a perpendicular from the arc to the middle of the chord is 1·35 in.; find the diameter of the saw. Ans. 16·35 in.

(15) An acre of ground, in the form of a circle, has a walk cut from it all round, 2 yds. broad, and the rest is grass; find the radius of the original circle, and the area of the grass-plot.

(1) Ans. 39·25 yds. (2) Ans. 4359·1584 sq. yds.

(16) The areas of two concentric circles are 165 yds. and 132 yds. respectively; find the breadth of the annulus between the circumferences.

Ans. 2·3 ft. *nearly*.

(17) It is required to construct, exactly in the middle of a circular area of 7 acres, a circular pond, which shall occupy one-third of the whole ground; find the radius of the pond, and the width of the ground left.

(1) Ans. 59·94 yds. (2) Ans. 43·89 yds.

(18) Find the area of the annulus formed by the super-position of a circle, whose diameter is 26·5 feet, on another circle whose diameter is 28·2 feet.

Ans. 73·034 sq. ft.

(19) An animal, tethered by a rope fastened to a stake in the straight hedge of a field, is allowed an acre of grass; what will be the length of the rope?

Ans. 55·5 yds. *nearly*.

(20) Two circles, each having a radius of 1 inch, intersect so that the circumference of each passes through the centre of the other; find the area which is common to both. Ans. 1·228 sq. in. *nearly*.

(21) Two circles touch one another internally, the radius of the larger circle being 2 in.; find the distance between the centres, when the area of the smaller circle is half that of the larger. Ans. ·5858 in.

(22) Two circles touch one another externally, the areas being as $2\frac{7}{9} : 1$; find the distance between the centres, if the smaller radius be 1 inch. Ans. $2\frac{2}{3}$ in.

(23) Find the measure of the angle at the centre of a circle which is subtended by an arc equal to the diameter. Ans. $114^\circ\ 35\frac{1}{2}'$, *nearly*.

(24) Supposing the diameter of the earth, as seen at the sun, to subtend an angle of 17·16″, employ the result of (23) to find the distance of the earth from the sun, the earth's diameter being taken as 8000 miles.

Ans. 96,160,800 miles.

(25) A man whose eye is 6 feet above the level of the sea can just see a small boat in the horizon; how far is the boat from him? Ans. 3·015 miles.

(26) In the last Example, how high must the man mount to see the boat at *twice* the distance?

Ans. 4 times as high, that is 8 yds.

(27) Shew how a circle may be described equal to the sum of *any number* of given circles. See 122, Part II.

(28) It is required to compare the numerical values of two given straight lines, and upon applying the process described in Prob. 15, the successive quotients are 1, 3, 6, 8; find the ratio of the lines corresponding to each of these quotients.

Ans. 1, $1\frac{1}{3}$, $1\frac{6}{19}$, $1\frac{49}{155}$.

(29) It is required to compare, as in Prob. 16, two given circular arcs, or the angles which they subtend at the centre; the quotients obtained are 2, 4, 1, 5; find the successive approximations to the true value of the larger arc, when the smaller one subtends at the centre an angle of 18°. Ans. 36°, $40\frac{1}{2}$°, $39\frac{3}{5}$°, $39\frac{21}{29}$°.

NOTE.

243. No intimation has been usually given in treatises on *Mensuration* of the actual methods by which the value of π is obtained; notwithstanding, it is expedient that the advanced student at least should be made acquainted with them, especially as there is no need to have recourse, as is commonly supposed, to *Trigonometry* for this end. Thus,

1st. *To shew that π lies between 3 and 4.*

In any circle inscribe a regular hexagon: then it is easily shewn that the side of the hexagon is equal to the radius of the circle, and therefore its whole perimeter

$= 6$ times the radius $= 3$ *diameters*. Again, circumscribe a square about the same circle, and it is easily seen, that the perimeter of the square $= 4$ *diameters*. But the circumference of the circle evidently lies between the perimeter of the inscribed hexagon and that of the circumscribed square; that is, it lies between 3 and 4 diameters, and π is the ratio $C : D$;

$$\therefore\ \pi \text{ lies between the numbers 3 and 4.}$$

2ndly. *To shew that π is equal to $3\frac{1}{7}$, or $\frac{22}{7}$, nearly.*

This is said to be the result obtained by Archimedes; and it is true that Archimedes was the first who undertook to compare the circumference of a circle with its diameter. His method was as follows:—From half the side of the *circumscribed* regular hexagon (whose numerical value in terms of the radius is readily found) he computed successively the half-side of the circumscribed regular polygons of 12, of 24, of 48, of 96, sides; always so, that the numerical values of the sides were *greater* than their true irrational values, but yet approximated closely to them. In this manner he found, that the ratio of the perimeter of the circumscribed regular polygon of 96 sides to the diameter of the circle, and therefore *à fortiori* the ratio of the circumference of the circle to its diameter, was *less than* the ratio of 14688 to $4673\frac{1}{2}$. Hence, $\frac{14688}{4673\frac{1}{2}}$ is *greater than* the ratio of the circumference of the circle to its diameter.

Next, he computed, in the same manner, the perimeter of the *inscribed* regular polygon of 96 sides, taking care that the numerical values of the sides were *less than* their true irrational value. He found that the perimeter of the *inscribed* regular polygon of 96 sides, and therefore *à fortiori* the circumference of the circle, had a *greater* ratio to the diameter of the circle than 6336 to $2017\frac{1}{4}$. Hence $\frac{6336}{2017\frac{1}{4}}$ is *less than* the ratio of the circumference of the circle to its diameter.

Now $3\frac{1}{7}$ is greater than $\frac{14688}{4673\frac{1}{2}}$, but approaches nearly to it;

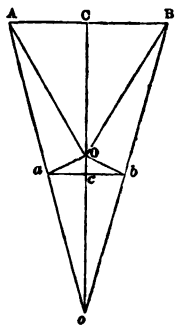
and $3\frac{10}{71}$ is less than $\dfrac{6336}{2017\frac{1}{4}}$, but approaches nearly to it;

$$\therefore \pi \text{ lies between } 3\tfrac{1}{7}, \text{ and } 3\tfrac{10}{71}.$$

This is the conclusion at which Archimedes arrived; and hence it is not correct to say, that he gave $\dfrac{22}{7}$ as the value of π. He simply gave $3\frac{1}{7}$, and $3\frac{10}{71}$, as the superior and inferior *limits* of the value of π; that is, he shewed that π is less than $3\frac{1}{7}$, but greater than $3\frac{10}{71}$.

3rdly. *To shew that π is equal to* $3{\cdot}14159$ &c.*

To do this we require first to solve the following problem, or some other to the same effect: viz. *Given the radii of two circles, one inscribed in, and the other circumscribed about, a given regular polygon, to find the radii of the circles inscribed in, and circumscribed about, a regular polygon* of the same perimeter, *but having double the number of sides.*



Let AB be a side of the given polygon, O the centre of the inscribed and circumscribed circles, OC perpendicular to AB. Join OA, OB. Produce CO to o, making Oo equal to OA. Join Ao, Bo. From O draw Oa, and Ob, perpendiculars to Ao, and Bo. Join ab cutting Oo in c.

Then since $Oo = OA$, $oa = Aa$;

and similarly $ob = Bb$.

Also,

$$AB : ab :: oA : oa :: 2 : 1;$$

$$\therefore ab = \frac{1}{2} AB.$$

Hence a regular polygon, whose side is ab, will have the *same perimeter* as that, whose side is AB, if the *number* of sides in the former be *double* the number in the latter. Also it is easily seen, that

$$\angle aob = \frac{1}{2} \angle AOB;$$

* The following proof is taken, with some slight alteration, from Sonnet's *Géométrie Théorique et Pratique.* Paris, 1853.

therefore, AB being the side of a regular polygon, OC the radius of its inscribed, and OA the radius of its circumscribed, circle, a regular polygon of twice the number of sides and of the same perimeter will have ab for its side, oc the radius of the inscribed, and oa the radius of the circumscribed, circle. It remains to find oc and oa in terms of OC and OA. Thus,

$$oc : oC :: oa : oA :: 1 : 2;$$

$$\therefore\ oc = \frac{1}{2}oC = \frac{1}{2}(Oo + OC) = \frac{1}{2}(OA + OC).$$

Or, if $OA = R$, $OC = r$, $oa = R'$, $oc = r'$,

$$r' = \frac{1}{2}(R + r) \ldots\ldots\ldots\ldots\ldots (1).$$

Again, since Oao is a right-angled triangle (by 72, Part I.)

$$Oo : oa :: oa : oc;$$

$$\therefore\ oa^2 = Oo \times oc = OA \times oc;$$

$$\therefore\ oa = \sqrt{OA \times oc},$$

$$\text{or } R' = \sqrt{R \times r'} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now, to apply this to the case before us, suppose, first, our polygon to be a square with a perimeter represented by 8; then it is plain that the radius (r) of the inscribed circle is 1, and the radius (R) of the circumscribed circle is $\sqrt{2}$, or 1·414213. Hence for a polygon of 8 sides, having the same perimeter, 8,

$$r' = \frac{1}{2}(\sqrt{2} + 1) = \frac{1}{2}(2{\cdot}414213) = 1{\cdot}207107,$$

$$\text{and } R' = \sqrt{\sqrt{2} \times \frac{1}{2}(\sqrt{2} + 1)} = \sqrt{1 + \frac{1}{2}\sqrt{2}} = \sqrt{1{\cdot}707107}$$
$$= 1{\cdot}306563.$$

Proceeding in the same way for a polygon of 16 sides, *with the same perimeter*, rad. of inscribed circle $= 1{\cdot}256835$, and rad. of circumscribed circle $= 1{\cdot}281457$; and so on, doubling the number of sides of the polygon

at each step. The following table shews the results, as far as it is necessary to go for our purpose:—

No. of sides of Polygon.	Rad. of inscribed Circle.	Rad. of circumscribed Circle.
4	1·000000	1·414213
8	1·207107	1·306563
16	1·256835	1·281457
32	1·269146	1·275287
64	1·272217	1·273751
128	1·272983	1·273367
256	1·273175	1·273271
512	1·273223	1·273247
1024	1·273235	1·273241
2048	1·273238	1·273239
4096	1·273239	1·273239

Hence it appears, that in a regular polygon of 4096 sides, whose perimeter is 8, the radii of the inscribed and circumscribed circles do not differ from each other by so much as ·000001; and as each circumference is less than 8 times its radius (by the 1st case), the difference of the two *circumferences* is less than ·000008, and therefore less than ·00001. *A fortiori* the circumference of either circle will not differ from the perimeter of the polygon which lies between them by a quantity so great as ·00001. So, then, we conclude that, with a near approximation to exactness, the circle, whose circumference is 8, has a radius equal to 1·273239.

$$\text{But } \pi = \frac{\text{circumf.}}{2 \times \text{rad.}} \quad (237),$$

$$\therefore \pi = \frac{4}{1{\cdot}273239} = 3{\cdot}14159 \text{ \&c.}^*$$

* As has been before stated, the value of π has been found, by other methods, correct to a much higher number of decimal places than the above; but it is seldom necessary to use a nearer approximation than 3·1416. Klügel, in his *Mathematisches Wörterbuch*, says that *Vieta*, by means of inscribed and circumscribed polygons, obtained the value of π correct to the 10th place of decimals, and the 1st edition of the book in which this very correct value is given was published as far back as A.D. 1579.

4thly. *To shew that* $\pi = \frac{355}{113}$, *nearly.*

The history of this remarkable approximation to the value of π is involved in some obscurity. It is certainly due to Peter Metius; for his son, Adrian Metius, in his *Geom. Pract.* (A.D. 1640), says that his father published this ratio in answer to the quadrature of Simon à Quercii, supposed to be Simon Duchesne; and it was done, he says, *Archimedeis demonstrationibus*, meaning, probably, by the inscribing &c. of polygons. Nothing further seems to be known respecting the ratio. But Professor De Morgan, a high authority on such subjects, has kindly furnished me with a clever conjecture of his own as to the *probable* method employed by Peter Metius. He thinks it likely that $\frac{355}{113}$ was only an appendix to Metius' result from the polygons, and not the result itself. That result would be 3·14159265; and having this before him, and moreover knowing that $\frac{3}{1}$ and $\frac{22}{7}$ are limits between which π lies, he tried the fraction $\frac{22m+3}{7m+1}$ with different values for m, until he hit upon $\frac{22 \times 16 + 3}{7 \times 16 + 1}$, which produces $\frac{355}{113}$, the fraction in question*.

It is to be observed, that $\frac{355}{113} = 3{\cdot}1415929$, &c., and therefore is correct to 6 places of decimals, and no further.

It is also easily retained in the memory from the circumstance that it is composed of the first three *odd* numbers in pairs, 113|355, taking the first three digits for the denominator and the remaining three for the numerator.

* The same result may now be easily obtained, by the method of '*Continued Fractions*,' from $\frac{31415926}{10000000}$. But this method was not in use in the days of Peter Metius.

SCALES.

244. When the linear dimensions of any surface have been obtained, and it is required to make a representation thereof; or, when the dimensions of any diagram of a surface have been obtained, and it is required to make another diagram of dimensions either larger or smaller than those of the original; it is clearly necessary to adopt some means whereby we may be sure that, whatever size we determine upon, the magnitudes of all the lines of the representation or diagram, which we are about to make, may bear a certain *uniform ratio* to those of the corresponding lines in the original.

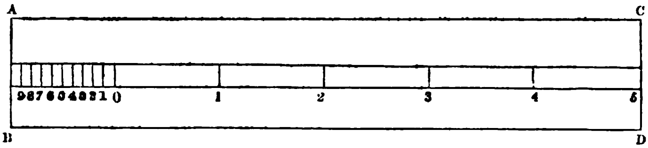
This process is termed *drawing to a Scale.* For this purpose the draughtsman either divides for himself a straight line on paper into such parts as will best suit his purpose, or procures an instrument so divided, that from it the various dimensions which have once been obtained by actual measurement, may be accurately transferred to his plan, according to some fixed proportion previously agreed upon.

Such an instrument is called a *Scale.* And it is usual in every such diagram, map, or plan, either to express in the margin the proportion which every line, or length, in it bears to some stated *unit*, or actually to *draw* at the foot of the diagram, map, or plan, *the scale* according to which it is constructed.

Thus, if it be written in the margin of a diagram, map, or plan, '*Scale, a yard to one-tenth of an inch,*' then every line, or length, in the diagram, map, or plan, which is measured by $\frac{1}{10}$th of an inch, actually means 1 yard; $\frac{2}{10}$ths of an inch mean 2 yards; $\frac{3}{10}$ths 3 yards; and so on. In this case we require both compasses, and the *Plain Scale* described below. But if the *Scale* itself be *drawn* at the foot of the diagram, map, or plan, then the compasses only are required, to enable us either to determine the length of any line already drawn, or to draw other lines in strict proportion.

245. A PLAIN SCALE, when it assumes the form of an *instrument,* consists of a thin flat rectangular piece of box

or ivory*, usually about 6 inches long, and $1\frac{3}{4}$ inches broad; and in its simplest form contains only two parallel lines on one of its faces, drawn in the direction of its length, and divided by small lines at right angles to the former, and at equal intervals of 1 inch, $\frac{1}{2}$ in., or some other unit of length agreed upon. Thus *ABDC* represents such an instrument, the two parallel lines on its face being divided into 6 equal parts, and the points of division marked 0, 1, 2, 3, 4, 5.



The length of each of the portions so formed may be taken to represent one mile, or one yard, or other unit of length; and for the subdivision of the unit the first of them to the left is divided into so many equal parts, that each shall represent one of the denomination next inferior to that of the assumed *unit*, or some convenient number of them. Thus, if the *Scale* be one of *feet*, the subdivisions will be *inches*, that is, *twelfths* of the unit: if the *Scale* be one of *yards*, the subdivisions will be *feet*, or *thirds* of the unit; if of *miles*, the subdivisions will be *furlongs*, or *eighths* of the unit, &c. For general purposes, however, it is most convenient to subdivide the unit into *tenths*, as in our diagram above.

When this *Scale* is used, and the unit on it is an inch, as for example in laying down lines, or lengths, *to a scale of a yard to an inch*, suppose we want to find the line corresponding to 4·8 yards; put one foot of the compasses upon the point in the *Scale* numbered 4, and the other upon the number 8 in the subdivisions of the unit; then it is clear, that there is intercepted between the points of the compasses a length of 4 units and 8 tenths, that is, 4·8. And by considering the unit on the

* Ivory, though commonly used, is a bad material for the purpose, since its length varies with moisture; box is much better.

scale as 1, or 10, or 100, or $\frac{1}{10}$, or $\frac{1}{100}$, the above intercepted length will represent 4·8, or 48, or 480, or ·48, or ·048, respectively.

Ex. 1. Represent 118 ft. in a *Scale* of a foot to one-tenth of an inch.

This quantity will measure on the *Scale* 118 tenths of an inch, or 11·8 inches, *i. e.* once the length of the *Scale*, (if its whole length be 6 inches,) and 5·8 inches more. So that, after taking the whole length of the *Scale* by the compasses, a second measurement must be taken, wherein one point will be exactly at the end of the *Scale*, on the figure 5, and the other upon the subdivision marked 8.

Ex. 2. What must one inch of the *Scale* represent, in order that ·18 feet may be represented upon it, by the distance between the large division marked 1, and the subdivision marked 8?

Here ·18 ft. $=\frac{1}{10}$th of 1 foot $+\frac{8}{100}$ths of 1 foot; hence, each of the large divisions must represent $\frac{1}{10}$th of 1 foot, and the small divisions $\frac{1}{100}$th of 1 foot; or, the *Scale* will be one-tenth of a foot to an inch, or 1 foot to 10 inches.

If the quantity had been ·018 feet, the *Scale* must be 1 inch to *one-hundredth* of a foot, or 100 inches to a foot.

N.B. *The subdivisions of the unit may be other than tenths.*

Thus, suppose them to be twelfths of an inch, each twelfth representing 1 foot; and let it be required to measure 43 ft. thereby.

Then, 43 ft. = 43 twelfths of an inch $= 3\frac{7}{12}$ inches, hence the compasses must embrace 3 units and 7 twelfths; or, one point must be upon the larger division marked 3, and the other on the subdivision marked 7.

Nothing less than 1 foot could be laid down from this *Scale*. Also any *large* number of yards and feet,

as 93 yds. 2 ft., must be converted into feet, viz. 281 feet, and this would give on the *Scale* 281 twelfths, or $23\frac{5}{12}$ in. And, since the *Scale* embraces only 6 units, or inches, the length $23\frac{5}{12}$ in. would be laid down by repeating the whole length of the *Scale* 3 times, and then taking $5\frac{5}{12}$ more, as the $3\frac{7}{12}$ was taken above.

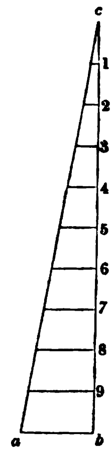
But if it is needful to carry the division to *hundredths* of an inch, so as to lay down a line whose measure consists of three places of figures, as 487, the above tenths must be further divided, each into tenths, or the unit into hundredths.

But if such divisions were made, few could count them. To enable us to make use of these minute subdivisions without confusion, we construct, or procure, what is called a

DIAGONAL SCALE.

246. It has been shewn in (172, Part II.) how to take any required portion as $\frac{1}{10}$th, $\frac{2}{10}$ths, &c....... of a *small* straight line; and if the small line be itself a tenth of any assumed unit, as 1 inch, then the tenths thereof will be *hundredths* of the same unit.

Thus, let *ab*, in the annexed diagram, be the tenth of the *unit;* from *b* draw an indefinite straight line at right angles to *ab*, and with any small opening of the compasses step along this line from *b* to *c*, dividing *bc* into 10 equal parts, and marking the points of division 1, 2, 3, 4, 5, 6, 7, 8, 9, as in the diagram.

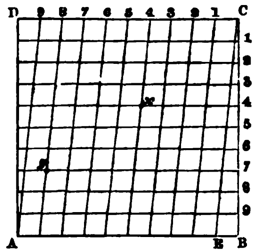


Join *ac*, and through the points of division let lines be drawn parallel to *ab*; these parallels intercepted between *ac*, and *bc*, beginning with the least, will therefore be

$$\frac{1}{10},\ \frac{2}{10},\ \frac{3}{10},\ \text{\&c. of } ab,$$

$$\text{or } \frac{1}{100},\ \frac{2}{100},\ \frac{3}{100},\ \text{\&c.......}$$

of the *unit*.

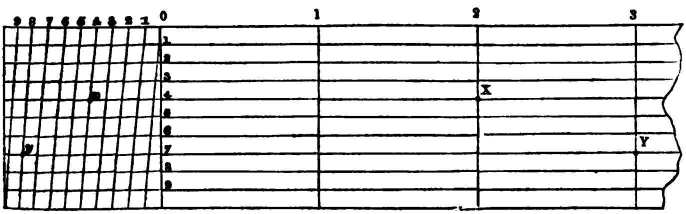


Let now AB be taken to represent the *unit*, and upon AB, as a base, construct a rectangle, $ABCD$; divide AB, BC, and CD, each into 10 equal parts; and let the points of division of CB, and CD, be separately numbered, 1, 2, 3, 4, 5, 6, 7, 8, 9, beginning in both cases from C. Then through the points of division of BC draw lines parallel to AB or CD; and join the points in AB with those in CD, as follows. Join C with the 1st point of division, E, in BA; join the 1st point in CD with the 2nd in BA; the 2nd in CD with the 3rd in BA; and so on, as is shewn in the diagram. Then the distance between each contiguous two of these diagonals*, measured on any of the former parallels, is $\frac{1}{10}$, or $\cdot 1$, since $AB=CD=1$; therefore $C4$, for example, $=\frac{4}{10}$, and hence

$$x4=\frac{4}{10}+\frac{4}{10} \text{ of } EB=\frac{4}{10}+\frac{4}{100}=\cdot 44.$$

Similarly, $y7=\frac{8}{10}+\frac{7}{100}=\cdot 87.$

And if this rectangle $ABCD$, with the first set of parallels only, be repeated longitudinally to the right, we have the ordinary *Diagonal Scale*, as below, in



* Called *diagonals* because, if *corresponding* points in AB and CD were joined, each would be the *diagonal* of a parallelogram.

which, for example, $Xx = 2{\cdot}44$, and $Yy = 3{\cdot}87$; and so on; the figure above Y being units, that above y in the same *diagonal* line *tenths*, and the figure in the same horizontal line as y being *hundredths*.

The above *Scale*, if extended to *ten* intervals, each equal and similar to the one just described, omitting the *diagonal* lines, will enable us to lay down the length of lines whose magnitude ranges from 1 to 1000, or from ·001 to 1.

Ex. Let it be required to lay down a line which shall represent the number 783. Here each unit of the *Scale* will have to represent 100, and 7 of them must be taken to make 700; the *diagonal* numbered 8, from 8 to 0, will give 8 tens, or 80; and the parallel in the triangle, marked 3, will give three-hundredths of the unit, that is, 3.

We, therefore, fix one foot of the compasses on the point where the parallel through the triangle numbered 3, meets the diagonal numbered 8, and then make the other foot to coincide with the point exactly under 7 in the division of units, and in the same parallel as the former point. Then it is clear, that the compasses embrace a length of 7 hundreds, 8 tens, and 3, or 783.

If the unit of the *Scale* represented 10 instead of 100, the line just taken would give the magnitude of 78·3; and if the unit represented 1, the line would be 7·83.

If the magnitude to be represented had been 1783, we should find the line which is measured by 783, as before, and then add to this the whole length of the *Scale*, that is, ten units. Or we should construct a *Scale* adapted for magnitudes which are expressed by 4 digits; as may easily be done.

And, as before, if the unit represented 10 or 1, instead of 100, the corresponding magnitudes of the line taken would be 178·3 and 17·83.

In order to lay down magnitudes from 1 to ·001, we must consider the whole length of the *Scale* to represent 1, and the *diagonal scale* will then give *thousandths*. And the aforesaid lines on the *Scale* would, with this unit, have represented ·783 and 1·783.

Ex. 1. Take from the diagonal scale the lines measured by ·123, and 6·07.

1st. For ·123, we take $\frac{1}{10}$ as the unit, and therefore the interval between two contiguous diagonal lines is $\frac{1}{100}$. Place one foot of the compasses where the parallel line, numbered 3 in the triangle, meets the diagonal numbered 2; and the other foot on the same parallel exactly below the number 1 in the division of units. We thus embrace $\frac{1}{10}+\frac{2}{100}+\frac{3}{1000}$, or ·123.

If we had taken 1, 10, 100, as our units, the same line would have represented 1·23, 12·3, 123.

2ndly. For 6·07, we take 1 as the unit; and then the interval between two contiguous diagonals is $\frac{1}{10}$; we therefore place one foot of the compasses where the parallel line, numbered 7 in the triangle, meets the diagonal marked 0, and the other foot on the same parallel exactly below the number 6 in the division of units.

If 10, 100, $\frac{1}{10}$, had been the units, the above line would have represented 60·7, 607, ·607, respectively.

Ex. 2. Take from the diagonal scale the lines measured by 1·025, and 187·2.

1st. For 1·025, we take $\frac{1}{10}$ as the unit, and therefore the distance between two contiguous diagonal lines is $\frac{1}{100}$. Place one foot of the compasses where the parallel line, numbered 5 in the triangle, meets the diagonal numbered 2, and the other foot on the same parallel exactly below or above the number 10 in the division of units, if the scale be continued so far *. We

* If the *scale* does not extend so far in the division of units, the whole number of units must be taken at twice.

thus embrace 10 units, each $\frac{1}{10}$, 2 hundredths, and 5 thousandths.

If we had taken 1, or 10, or 100, as our unit, the same line would have represented 10·25, 102·5, 1025, respectively.

2ndly. For 187·2 we take 10 as the unit, and place one foot of the compasses at the intersection of the parallel, numbered 2 in the triangle, with the diagonal numbered 7, and the other foot exactly above or below the number 18 in the division of units, if the scale be continued so far*.

If the unit were 1, or 100, or $\frac{1}{10}$, the same magnitude would have represented 18·72, 1872, 1·872, respectively.

247. Conversely, if it be required to measure the length of a proposed line in any diagram, or plan, by a given *scale*, we open the compasses so as to embrace the whole line, and then place one foot upon one of the great *unit* divisions of the *scale*, marked 1, 2, 3,......, so that at the same time the other foot may fall somewhere among the figures which mark the *diagonal* divisions. If the second foot does not at once fall upon an exact point of division, let the former foot be moved along the cross line in which it was placed, until the other foot falls upon the intersection of one of the diagonal lines with one of the parallels which run lengthways on the scale, taking care that *both* feet of the compasses are on the *same parallel*. Then in the number which indicates the measure of the proposed line, the highest denomination will be the number in the units division opposite to the first foot of the compasses; the second figure will be the number at the extremity of the *diagonal* on which the other foot rests; and the third, the number of the parallel in the triangle which, produced both ways, passes through both feet of the compasses.

Thus, if the measure of the line be 753, or 75·3, we shall have one foot of the compasses in the cross line of

* See note p. 267.

the unit divisions marked 7, and the other at the intersection of the 5th diagonal, with the third parallel running lengthways on the scale.

The *unit* in the former line is, of course, *tenfold* that in the latter.

248. The *Diagonal Scale* is not restricted to *decimal* measurement, but may be constructed to measure the second and third next inferior denominations of *any given unit,* whether they be decimal, duodecimal, or any other given fractional parts of the primary unit. Thus, supposing the *scale,* as before, composed of a series of equal rectangles, placed so as to form one whole rectangle of the same height, the length of each rectangle in the series being either equal to, or considered to represent, the given unit, we divide the *length* of the first rectangle into as many equal parts as the next inferior denomination is contained in the primary unit. Then we divide the *height* of the same rectangle into as many equal parts as the third inferior denomination is contained in an integer of the second; and by drawing the diagonals and parallels, as before, according to these subdivisions, we have the *Diagonal Scale* required. Thus,

Ex. 1. To construct a *diagonal scale* six inches long, of 9 feet to an inch, to measure inches.

Here the assigned length of the scale being 6 inches, we have the primary division, or unit, 1 inch: the next inferior denomination $\frac{1}{9}$th of the primary—and the third denomination $\frac{1}{12}$th of the second. So we divide the whole scale into 6 equal rectangles, the length of each being 1 inch, but representing 9 feet. Then we divide the *length* of the first rectangle into 9 equal parts, so that each part is the ninth of an inch, but represents a *foot.* Then we divide the height of the same rectangle into 12 equal parts; and draw the several diagonals and parallels in the usual way, and thus we have a measure for *twelfths* of a *foot,* that is, for *inches*, as required.

Ex. 2. To construct a diagonal scale of 1 mile to an inch to measure *perches.*

Make the primary divisions each 1 inch. Divide the primary into 8 equal parts, so that each part represents 1 furlong. Also divide the height of the scale into 40 equal parts; and complete the construction as usual. Then the triangular parallels will give us a measure of fortieths of a furlong, that is, for perches.

THE PLOTTING SCALE.

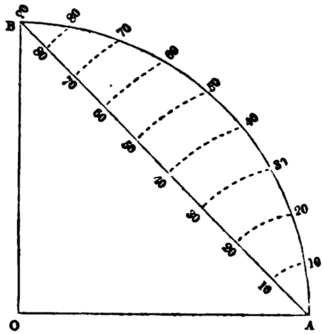
249. This is a long, thin, flat instrument usually made of box, whose sides are perfectly straight, and a portion of whose upper surface is bevelled off at both sides to a very fine edge, its under surface remaining quite flat. Upon the bevels on both sides, a few particular plain scales are pointed off, so that their divisional lines, being drawn down close to the hair-line edges, and thus ending, as it were, upon the very paper to which they may be applied, scale measurements can be plotted or marked off therefrom, at once, without the aid of the compasses. The entire length of this scale is usually divided into two equal parts, by a line drawn right across its bevelled or upper surface *at right angles* to both edges; and this line may be used for drawing perpendiculars to any given line to which the edge of the scale may be applied for that particular purpose. For land-surveyors, the plain scales marked upon the bevels are scales of chains and links, so many chains to the inch; for architects, scales of feet and inches, so many feet to the inch; and so for other draughtsmen according as they find certain particular scales convenient in their particular practice.

An accurate suitably graduated *plotting-scale* is one of the most useful instruments in the hands of a draughtsman, for by its means he can draw straight lines, or lines according to scale; or measure lines drawn according to scale; erect perpendiculars, or draw the perpendiculars of already constructed triangles, and all without the aid of the compasses.

SCALE OF CHORDS.

250. On the reverse side of the *Diagonal Scale*, various scales are often given, of which one of the

most useful is the *Scale, or Line, of Chords*. This scale is constructed thus: $\angle AOB$ is a right angle; with centre O, and any convenient radius describe the arc AB of the quadrant AOB: and draw the chord AB. Divide the arc AB into 9 equal parts, so that each part is the arc of 10 degrees; and mark the divisions from A successively 10, 20, 30...80, 90. Then with centre A, and distances equal to the chords of 10, 20, 30, &c. degrees, set off with the compasses on the straight line AB each of these several chords, and mark their termini 10, 20, 30, &c. signifying that A 10, measured on the straight line AB, is the chord of 10 degrees, A 20 the chord of 20 degrees, and so on. The straight line AB thus divided is the *Line of Chords* required.



If each arc of 10 degrees be again subdivided into 10 equal parts, then, by proceeding as before, the chords of all arcs, in degrees, from 0° to 90°, may be transferred to the scale AB, and thus we have a measure *on the scale* of the chord of every arc from 0° to 90° in the circle whose radius is A 60, since the chord of 60° is always equal to the radius.

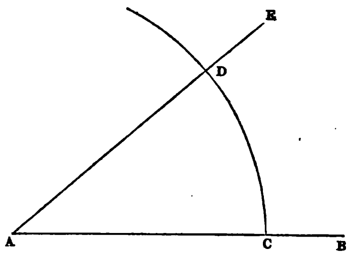
The *Line of Chords* serves two purposes, either to lay down an angle given in degrees, or to measure an angle already laid down.

1st. To lay down an angle, suppose, of 35°.

From the *line of chords* on the scale take off with the compasses the chord of 60°. Then, with this length for radius, and the point which is to be the vertex of the angle for centre, describe an arc. Take from the scale the chord of 35°, and with this opening of the compasses place one foot on the arc, and mark where the other foot also meets the arc. Join these two points in the arc with the centre, and we have the angle required.

If it be required to set off the angle, so that a proposed given straight line shall be one of the bounding lines which form the angle, then it is obvious that the first foot of the compasses must be placed upon the intersection of the arc with the given line.

Thus, if AB be a given straight line, and it is required to draw another line making an angle of 35° with the former, with centre A, and radius 60 from the *line of chords*, describe an arc cutting AB in C; then take 35 from the same scale, and set it off from C to D. Join AD, and CAD is an angle of 35°.



If the angle to be laid down be *obtuse*, since the *scale of chords* does not go beyond 90°, the angle must be divided into two, viz. 90°, and the excess above 90°; each of these being laid down separately, but contiguous, the sum of the two will plainly be the angle required.

2ndly. To measure an angle already laid down; let BAE be the given angle. With centre A, and radius 60 from the *line of chords*, describe an arc intersecting AB and AE in C and D. Then, with the compasses, take the length of the chord of CD, and apply it to the *line of chords* with one foot upon 0; the number which coincides with the other foot will be the numerical measure of the proposed angle in degrees.

251. It has been shewn that by means of the *Plain*, or *Diagonal, Scale*, diagrams or plans are *reduced* in any required proportion; and it is to be understood that similarly they may be *enlarged*, if needful, in any given proportion. Such results may also sometimes be conveniently obtained by means of the *Proportional Compasses*, or the *Pantagraph*, described in (177 and 178, Part II.). But *in all cases* the learner must bear in mind that the reduction or enlargement in question is according to

linear measure; that is, corresponding *lines*, and not *areas*, are *in the stated proportion*. Thus, if any diagram, or plan, in the form of a polygon, is to be reduced from, suppose, the scale of a yard to an inch, to the scale of a yard to one-fourth of an inch, a *similar* polygon is constructed, in which each *side* is one-fourth of the corresponding side of the former polygon; but, since by (92, Part I.) the *areas* of similar polygons are to one another as the *squares* of any homologous sides, the area of the new polygon is not $\frac{1}{4}$th, but $\frac{1}{16}$th, of the area of the given polygon.

In like manner, if in any diagram or plan, which is to be reduced or enlarged according to a certain scale, a *circular area* is found, the reduction or enlargement is effected by taking the *radius* according to the reduced or enlarged scale, and describing such an arc as will subtend the same angle at the centre. The *circular arcs* in the two diagrams will thus be in the stated proportion; but the *areas*, as in the case of rectilineal figures, will be to each other as the *squares* of the radii (see 93, Part I.).

QUESTIONS AND EXERCISES K.

[In the following Exercises the *Scale* is *decimally* divided, except when it is otherwise stated.]

(1) Explain clearly the object of *Scales* in general; and exhibit the simplest form of *Scale* in common use.

(2) Point out the difference between a *Plain Scale* and a *Diagonal Scale*, both as to form and power.

(3) What is the greatest error which can arise from measuring with an ordinary *Diagonal Scale*?

Ans. Less than ·01.

(4) State the position of the feet of the compasses on a *Diagonal Scale*, when they include a length measured by the number 3·29.

(5) Explain the operations of laying down, from the same *Diagonal Scale*, the dimensions represented by the numbers, 187·5, 245·3, and 110·5.

(6) What alteration of the unit of measurement is necessary, to enable us, *by the same interval* between the

feet of the compasses, to indicate ·329, 3·29, 32·9, and 329.

(7) On a *Diagonal Scale*, which is a foot in length, and divided into ten equal parts, how many inches and decimal parts of an inch would measure the several numbers, 327, 453, and 35?

Ans. 3·924 in., 5·436 in., ·42 in.

(8) What is the length of a *Scale*, divided into 20 units, on which the number 18·5 measures 3 inches and 7 tenths? Ans. 4 inches.

(9) If the base of the diagonal compartment of a *Scale* be divided into 8, and the height into 10, equal parts, how many of the lowest measures on the scale are contained in one of the highest? Ans. 80.

(10) Suppose each of the first subdivisions of the primary unit in the scale (Ex. 9) to represent 3 inches, what will be the arithmetical measures of its smallest, and of its primary, divisions?

(1) Ans. $\frac{3}{10}$ths of an inch. (2) Ans. 2 feet.

(11) What was the *Scale* used in the construction of a plan, upon which every square inch of surface represents a square yard? Ans. Scale of 3 feet to an inch.

(12) What is the *Plain Scale* on which a length of 4 ft. 10 in. measures exactly $4\frac{5}{8}$ inches?

(13) What is the *Diagonal Scale*, measuring inches, upon which 4 ft. 10 in. is represented by $2\frac{5}{12}$ inches?

Ans. Scale of 2 feet to an inch.

(14) The ratio of one *Scale* is 2 : 1, and of another 3 : 1, on which will a given length measure *most*?

(15) In what ratio will the *scale* length of a given line, as measured on a *scale*, whose ratio is 4 : 1, exceed that of the same line as measured on another *scale*, whose ratio is 5 : 1? Ans. $7\frac{1}{2}$: 6, or 5 : 4.

(16) A draughtsman laid aside an unfinished plan, and after a while resumed his work, but found that he had forgotten the *scale*. How will he proceed to recover the lost *scale*?

(17) A Land-Measuring Chain being 22 yards in length, make a *diagonal scale* of two chains to an inch, to shew feet.

(18) The chain, as before, being 22 yards, if a *scale* of 10 chains to an inch be taken, what will be the measure in acres of each square inch on the plan?

Ans. 10 acres.

(19) If each square inch on the plan were to represent 1 acre of surface, what would the *scale* be?

Ans. $\sqrt{10}$ chains to an inch.

(20) If a *Scale* be taken of 2 perches to an inch, and the *base* of the diagonal compartment be divided into 11 equal parts, whilst the *height* is divided into 12 equal parts, what will the smallest subdivision measure?

Ans. 3 inches.

LAND-SURVEYING.

252. One chief use of '*Geometry combined with Arithmetic*' consists in the mapping, and measuring, of land, called '*Land-Surveying*'.

The art of '*Land-Surveying*' includes two branches:

1st. The *laying down* on paper a representation, or map, of an estate or parcel of ground to be surveyed.

2nd. The measuring in known units, as in square yards and feet, or in acres, roods, and perches, &c., the content or area of the land proposed.

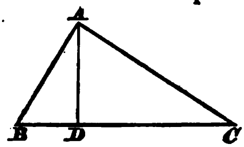
The second operation can be performed without the first, that is, without producing an exact plan, or map; for we may draw a rough sketch of the land, and if certain linear measurements be correctly taken, we may then *calculate* the area without any further reference to the plan, except as to its general outline. Or, we may make the plan accurately correct *to any given scale*, and then measure the area *from the plan*, according to the methods

given before for measuring plane areas of any form, rectilineal, or circular.

Both the above methods will be here exhibited, and examples worked, with a view of teaching the *principles*, but not all the practical details, of *Land-Surveying*.

253. *To map, and measure, any small piece of land bounded by straight lines, and considered as a plane surface.*

1st. Let the plot be a triangle, as ABC, which can be traversed in any direction.



Measure with a tape, on the ground, each of the sides AB, BC, CA.

Draw upon the paper a line, in any convenient direction, and on it lay off, *by a scale*, a length BC representing the arithmetical magnitude of the longest side BC. Next, from the same scale take off the measured distances represented by AB, AC; and with these as radii, from centres B and C, describe small intersecting arcs, to fix the true position of A. Then join AB, AC, and ABC will be a correct map of the proposed plot of ground.

Next, to *measure* the plot, we might pursue the method given in (230), for finding the area of a triangle in terms of its sides, and then no *plan* is actually required; but this method often involves rather heavy calculation. It is better therefore, after making a correct plan, to draw a perpendicular AD on BC from the point A, by one of the methods given in Part II. The measure of AD must then be taken from the *same scale* as that by which BC was laid down, and the area of the triangle will then be equal to half the product of BC and AD.

N.B.—If any means present themselves for determining *on the ground* the length of the perpendicular AD, it is obvious that there will be no occasion to measure in addition any of the boundary lines, except the longest, BC.

Also, if the plot be in the form of a *right-angled* triangle, it will only be necessary to measure the two sides

containing the right angle. The area will be half the product of those two sides.

2ndly, Let the plot be a *parallelogram*. Then, since it can be divided into two equal triangles by either of its diagonals, the mapping and measurement will be as before, except that when the sides of ABC have been laid down to the proposed scale, the remaining sides of the parallelogram are found by drawing lines parallel to AC, AB, respectively. And when the perpendicular AD has been determined as before, the area will be equal to the product of AD and BC.

If the parallelogram be *rectangular*, the perpendicular will coincide with one of the smaller sides, and the only measurements required will be those of any two adjacent sides.

Of course, if the plot be a *square*, it will only be necessary to measure a single side (222).

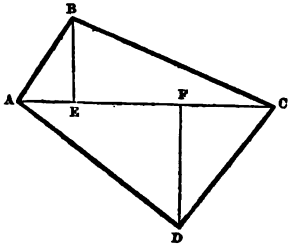
3rdly, Let the plot be a *trapezium*, that is, a quadrilateral with two of its sides parallel.

Measure one of the two sides that are not parallel and on any convenient line on the paper lay off, as before, *by a scale*, a length representing the arithmetical magnitude of that line. If the parallel sides be at right angles to this, draw two lines at right angles to the above base line from its extremities; measure the parallel sides, and lay off *from the same scale* their magnitudes upon these perpendiculars, join their extremities, and the plot is correctly mapped.

The *area* also is found, according to the method of Prob. 9, p. 217, by multiplying the numbers representing the magnitude of the base line and half the sum of the parallel sides.

But if the parallel sides be *not* at right angles to either of the other sides, then the area will be the product of half the sum of the parallel sides and the perpendicular distance between them; or it may be thought fit to treat the question as one of a *general* quadrilateral figure.

4thly. Let $ABCD$ represent the general outline of any quadrilateral plot of ground, which it is required to map and measure, and which, suppose, can be traversed in any direction.

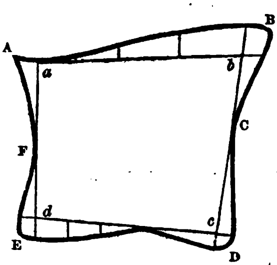


Then it can plainly be divided into two triangles by the diagonal AC. We have only therefore to proceed to lay down each of the triangles ABC, ADC, *to the same scale*, as in the first case, and measure the perpendiculars BE, DF, as there shewn. The area required will be equal to the product of the numbers representing AC the diagonal, and half the sum of the perpendiculars BE, DF.

5thly. If the proposed plot of ground be bounded by *more* than four straight lines, it must be divided into convenient triangles by means of diagonals; and the areas of the several triangles will obviously together make up the area required.

254. When, however, any of the boundary lines of a plot of ground proposed to be measured are not *straight*, we may adopt either of the two following methods of overcoming the difficulty.

We may draw straight lines as near to the boundaries as convenient, whether within or without the area, so as to indicate the general direction of the fences, as $abcd$, in the plot $ABCDEF$. The area of $abcd$ will be found, as described in the last article, and the small areas excluded are computed by measuring at every turn in the fence the perpendicular distance thereof from the main line. These perpendiculars are termed *Offsets*. The small areas included between the offsets, the fences, and lines ab, bc, cd, da, will be either triangles,

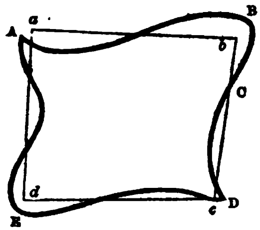


trapeziums, or rectangular parallelograms, which can be computed according to the directions already given, and their *sum added* to the area of the rectilineal figure *abcd*.

If it had been more convenient to draw the lines *ab*, *bc*, *cd*, *da*, *outside* of the proposed plot, the small areas between the true and assumed boundaries would have to be *subtracted* from that of the rectilineal figure *abcd*.

Another method, more simple than the one just described, and which may be called the '*give-and-take*' method, has already been noticed in p. 218.

By this method, as shewn in the annexed diagram, the plot being the same as before, instead of drawing the lines *ab*, *bc*, *cd*, *da*, either entirely in, or entirely out of, the real plot, they are drawn so intersecting the fences, that the parts excluded balance, as nearly as can be guessed, the parts included; and the process is at once reduced to the simple case of measuring the rectilineal figure *abcd*, as in (253).

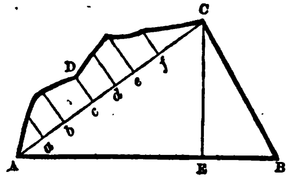


N.B.—When the small perpendiculars, or offsets, are very short, they are measured with a staff, called an *Offset-Staff*, which should be divided in the same manner as the measuring tape, but with small subdivisions, and should be numbered on two sides, but from different ends. For *short* offsets that can be measured with this staff, it will be sufficient to tell *by the eye* where the perpendiculars from the points at the extremities of the offsets meet the main line of measurement. But where the distances are much larger, we may determine the perpendicular direction of the offset, as in (253).

255. *To find the area of a plot of ground, of which the boundaries are not wholly rectilineal.*

1st. Let the plot have three sides, of which two are, or may be taken to be, straight lines, and the third irregular and curved as $ABCD$, in the diagram, where ADC is the only irregular boundary.

Join AC; and proceeding from A or B, measure AB,



and CE, the perpendicular from C upon AB, with a *Tape* divided into yards, and tenths of a yard.

Suppose $AB = 53{\cdot}9$ yards;
$CE = 28{\cdot}6$ yards.

If only the area be required, and not the plan, BC need not be measured. But if it be not convenient to measure CE, then all the three sides of the triangle ABC must be measured, and its area found by (230).

Next, to obtain the area of the irregular piece between AC and ADC, proceed along AC, and note every point in the fence ADC where there is any material change in its direction, and measure the perpendicular distance of that point from the line AC with the *offset-staff*, or *tape*. The small portion of the fence between every two contiguous offsets may be considered as a straight line; and the several areas, enclosed between the fence, AC, and the offsets, will be trapeziums, except the first and last, which will be triangles.

Let a, b, c, d, e, f, be the points in AC, where the offsets meet it, and suppose $Aa = 6{\cdot}4$, $ab = 5{\cdot}2$, $bc = 8$, $cd = 5{\cdot}6$, $de = 5{\cdot}6$, $ef = 5{\cdot}2$, $fC = 11{\cdot}8$, all expressed in yards.

Also, suppose the lengths of the several offsets at a, b, c, d, e, f, to be 6, 7·8, 5·8, 7·5, 7·8, 5·6, respectively. It will be most convenient to compute the *double* of the areas of the several trapeziums and triangles, and halve the total result.

			sq. yds.
Double the area upon	$Aa =$	$6{\cdot}4 \times 6 =$	38·4
....	$ab =$	$5{\cdot}2 \times 13{\cdot}8 =$	71·76
....	$bc =$	$8 \times 13{\cdot}6 =$	108·8
....	$cd =$	$5{\cdot}6 \times 13{\cdot}3 =$	74·48
....	$de =$	$5{\cdot}6 \times 15{\cdot}3 =$	85·68
....	$ef =$	$5{\cdot}2 \times 13{\cdot}4 =$	69·68
....	$fC =$	$11{\cdot}8 \times 5{\cdot}6 =$	66·08

$\therefore$ Sum of these areas $= 514{\cdot}88$

or area of $ACD = 257{\cdot}44$

$$\text{Also area of } \triangle ABC = \frac{53{\cdot}9 \times 28{\cdot}6}{2} = 770{\cdot}77$$

$\therefore$ the whole area $ABCD = 1028{\cdot}21$

If there be more than *one* crooked boundary, the measurement of a second irregular plot must be taken precisely as above.

And if the area to be measured is bounded by more than three sides, that is, if it partake not of the general form of a triangle, it may be divided into convenient triangles by means of diagonals, as mentioned at the end of (253).

But if any one of the irregular sides can be readily reduced to a straight one by means of the '*give-and-take*' principle, as exemplified in (254), of course the calculation by *offsets* of the irregular portion for that side will be done away with.

It must also be borne in mind, that the method by *offsets*, when the boundary is curvilinear, can only give an *approximation* to the true area; but yet the error can be diminished *as much as we please* by increasing the *number* of the *offsets*.

256. In the last article we have seen that a great many notes had to be made of the lengths, either of the sides of the triangle, or other measured distances. It is therefore advisable that these notes should be *registered* in some systematic manner, in order that either the surveyor himself, or some other person, may *at any time* from the inspection of the notes, recognise the general plan of the ground, and compute its area. The book in which these notes are thus conventionally registered is called a

FIELD-BOOK.

Each left-hand page of the book is divided into three columns, (the right-hand page being left blank for remarks); the central column is used for recording the lengths of the *main lines* measured, whether they be sides of triangles, or diagonals of trapeziums, or of other irregular figures. The left-hand column is used for perpendiculars lying to the left of any of these main lines; the right-hand column for perpendiculars lying to the right. It is usual to begin at the *bottom* of the central column, and work upwards. The *field-book* in the case of (255) is here given, which shews the arrangement adopted. Θ means *station;* the number at

the bottom of the central column gives the length of the first measurement, BE; the next of BA, formed by adding EA to BE: the number to the right denotes the length of EC; those to the left of the upper column denote the lengths of the offsets at a, b, &c.;

Perps. on left.		Perps. on right.
	47·8	to C.
5·6	36·0	
7·8	30·8	
7·5	25·2	
5·8	19·6	
7·8	11·6	
6·0	6·4	
From	⊙ A	
	53·9	to A.
	15·3	28·6
From	⊙ B	go West.

and the numbers in the upper compartment of the central column, beginning from the bottom, are the distances Aa, Ab, Ac, &c. measured from A to the several points, at which the offsets are taken, each distance being placed exactly in a line with the particular offset belonging to it.

When the piece of land to be surveyed is of considerable size, it is not convenient to use a *tape*, but a particular kind of chain which we now proceed to describe.

THE CHAIN.

257. The CHAIN used by surveyors for measuring land, and called *Gunter's Chain*, consists of 100 equal links of iron, with a handle at each end for the surveyor and his assistant. Its length is 4 poles, or 22 yards; and therefore the length of each link $=\frac{22 \text{ yds.}}{100}$ or ·22 yds. or ·66 ft., or 7·92 inches; that is, very nearly 8 inches. This length of 22 yds. is chosen, because it is most convenient for measuring an *acre*, which is 220 yards, or 10 chains,

long, and 22 yds., or 1 chain, broad. Hence, an acre is equal to 10 square chains, or 100,000 square links. Consequently, if the linear dimensions of a field be expressed in links, and its area thence obtained in square links, this value, when divided by 100,000, will be expressed in *acres;* that is, if *five* places be pointed off as a decimal, the result will be *acres*, and decimal parts of an acre, which decimal can be reduced to roods and perches.

Exactly in the middle of the chain is a piece of brass easily distinguishable; and from each end to the middle there is at every interval of 10 links a piece of brass, having *one* notch at 10 links, *two* at 20, &c. The object of this arrangement is to enable the surveyor to measure from either end with as little *counting* as possible.

258. *To measure a straight line with the chain.*

Let the surveyor place one end of the chain at one extremity of the line, and let an assistant apply the chain on the ground in the direction of the proposed line; then if the length of the line be less than one chain, it will be expressed in links, or hundredths of a chain. If the line be less than half a chain, the links are readily counted from the former end, or if the length be nearly 50 links, as 43, we may count the defect from 50, viz. 7, and so obtain the 43. If the line be more than half a chain, the excess above 50 may in like manner be readily counted, and added to the 50.

In measuring a portion of very valuable land, if there be not an exact number of links in any dimension, a link may with tolerable accuracy be divided by the eye into two equal parts, and the portion so counted will occupy the third place of decimals, or thousandth of the unit, where the unit is a chain. Thus the length of a line measuring $83\frac{1}{2}$ links would be written, ·835 of a chain.

In practice, *with the Chain*, no length less than *half a link* is ever noticed, but if the plot be a small portion of building land, in which a greater degree of accuracy than within half a link is required, it will be preferable to use the *Tape*, marked with minute subdivisions.

Again, if the length of the required line be more than a chain, let its position be indicated by two or

more poles placed vertically in the ground at convenient intervals, and so arranged, that they shall appear but as one pole to the eye of an observer at either end of the line. The line may then be readily measured by two persons walking and applying the chain successively from one end to the other, care being taken,

1st. To preserve the direction marked out by the poles.

2nd. To be sure, in applying the chain, that it is always straight and pulled tight.

3rd. To note down every time the entire chain has been applied; which is done by means of ten small arrows, or pins, of strong iron wire, about 18 inches long, and pointed at one end to stick in the ground. The assistant leaves one end of these in the ground at the end of each chain, which is taken up by the surveyor, when he arrives at it.

If the line does not consist of an exact number of chains, the portion of the last chain must be reckoned as above.

The following are specimens of the simplest cases that can occur of the measurement of land by the use of the *chain* alone.

Ex. 1. Find the number of acres in a rectangular field, whereof the length and breadth are respectively 25, and 17, chains.

$$\begin{array}{r} 25 \\ \underline{17} \\ 175 \\ \underline{25} \\ 425 \end{array}$$

425 sq. chains $=42{\cdot}5$ acres, or $42\frac{1}{2}$ acres.

Ex. 2. Let the dimensions be 35 chains 72 links, and 24 chains 8 links.

We have before shewn that if the dimensions be expressed in links, and thence the area be obtained in square links, we can convert the result into acres by pointing off

5 decimal places, and reducing the decimal part to roods and poles.

Hence we have

$$\begin{array}{rl}
\text{Length} = 35 \text{ chs. } 72 \text{ links} = & 3572 \text{ links,} \\
\text{Breadth} = 24 \text{ chs. } 8 \text{ links} = & 2408 \text{ links,} \\
\hline
& 28576 \\
& 142880 \\
& 7144 \\
\hline
\text{Area} = & 86{\cdot}01376 \text{ Acres} \\
& 4 \\
\hline
& {\cdot}05504 \text{ Roods} \\
& 40 \\
\hline
& 2{\cdot}20160 \text{ Poles}
\end{array}$$

and therefore the field contains 86 a. 0 r. $2\frac{1}{5}$ p. nearly.

If one dimension had contained an exact number of chains, it would still be well to reduce it to links, and proceed as before. Thus, if the dimensions were 15 chains, 23 links, and 12 chains,

$$\text{the area} = 1523 \times 1200 \text{ sq. links,}$$
$$= 18{\cdot}276 \text{ acres} = 18 \text{ a. } 1 \text{r. } 4{\cdot}16 \text{p.}$$

259. Having now shewn how to use both the measuring *Tape* and *Chain*, in finding the areas of plane surfaces, bounded either by straight or curved lines, we may notice that, in taking measurements for the construction of maps or plans, or for estimating the area of portions of land, we have often other difficulties to contend with; as,

I. *Inequality of surface*, requiring to be reduced to some level agreed upon.

II. *Inaccessibility*, where we can only go *round* the area, as in the case of a morass, or forest; or, where an object is separated from the observer by a stream, or other impediment.

And both these difficulties may be combined, as well as irregularity of outline, which has been treated of before.

Sometimes, also, it may be required, as in the construction of a railway, to make an exact representation of a vertical section of the country, extending from any one point to any other point, as from the level of the sea at Hull to that at Liverpool. And the outline of this section will be an irregular line, drawn on the earth's surface from the one point to the other; whilst below it is an *horizontal* line, which is supposed to be drawn between the two given points, and which would, of course, fall below the higher ground in the route. Such a correct representation would indicate, by successive risings and fallings, the exact variations of hill and dale, met with in passing from point to point, and the exact depths below the surface at which an horizontal* line from Hull to Liverpool lies through every point in its course.

This horizontal line, to which the successive points on the surface are referred, is termed the *datum* line.

In a moderately level country even the greatest height of any portion of the irregular line above the *datum* line is small compared with its length; and consequently the variations in height, and the inclination, of successive portions will hardly be manifest to the eye. We may therefore *multiply* the height of every point in the surface above the *datum* line by some multiplier; and the section thus altered will convey a correct idea of the truth; for though none of the lines are actual representations of the inclinations and altitudes, yet they will all bear the same ratio to one another that the actual lines do.

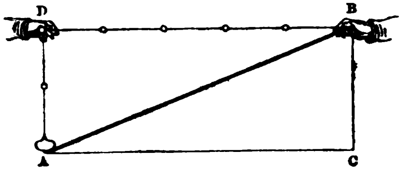
We shall, however, examine only such simple cases as serve to illustrate the *principles* of these investigations, and at the same time describe the instruments employed in the treatment of such simple cases.

260. In finding the *area* of a piece of ground, which is *not level*, we must not measure its actual surface, but the *horizontal* surface which would be seen, if all the inequalities were removed.

For, as we value land, only for what can either grow, or be built, upon it, and as all buildings are erected, and

* Of course a long line apparently horizontal will partake of the curvature of the earth, just as a line on the surface of the sea does.

products grown, only perpendicularly to the level of the sea, and not to any particular surface, as a hill-side, it is clear, that if we have to estimate the length of *AB*, then, for all practical purposes *AC* is the true length. And the point *C* is found by dropping *BC* perpendicularly to the horizontal line *AC*; or, in practice, the chain *DB* is stretched horizontal, and a portion of it allowed to hang vertically down at *A*. Or, one of the iron pins, used by surveyors may be dropped vertically from *D*; and thus we note the elevation of *B* and *D* above *A*, and the length of *AB* estimated horizontally.



AC, or *DB*, is technically called the *projection* of the line *AB* on a horizontal plane: and either of the horizontal planes in which *DB*, or *AC*, lies, is called the horizontal projection of the plane or surface in which *AD* lies. Such a projection would be the appearance presented in a bird's-eye view taken from a very great height.

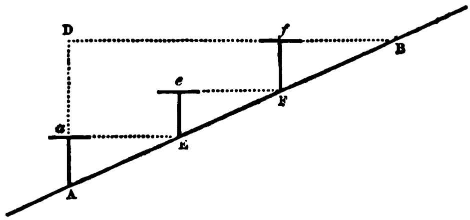
If the line *AB* be too long to admit of the chain being stretched over it, as in the last figure, we must employ an instrument termed

THE LEVEL.

261. The Level, in its simplest form, consists of an upright wooden rod or staff of any convenient height, say from 4 to 5 feet, at the top of which is placed a small bar at right angles to the staff. If this rod be pointed at the lower end, and fixed vertically in the ground by means of a plumb-line, so that the cross bar is horizontal, an eye looking along its length in the direction of a piece of rising ground will meet in its line of sight that point in the ground, which is on the same horizontal line as the eye, and will therefore shew, that the rise from the observer's station to the said point is equal to the height of the staff.

But if the rise of the ground, within a convenient distance for observation, is not equal to the height of the

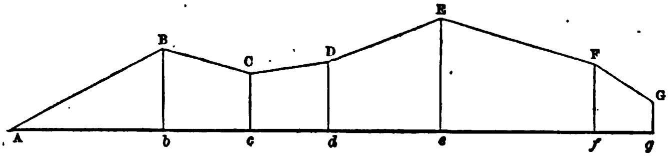
staff, a second staff marked with small divisions carried by an assistant must be placed opposite to the observer; then the *difference* between the height of the point on the second staff where the line of sight meets it, and the height of the staff at the observer's station will of course be the difference of level of the two places. It is obvious, that, when the ground rises in front of the observer, the height of the first staff will be greater than that of the observed point in the second staff; and when it falls, it will be less.



If therefore this instrument be successively placed at A, E, F, where there is a continuous rise, as from A to B, the whole perpendicular rise AD is found by adding the sum of the successive rises Aa, Ee, Ff, &c. And since we can thus measure AB more conveniently than DB, we obtain by (45, Part I.), $DB = \sqrt{AB^2 - AD^2}$.

A more accurate form of instrument consists of a tripod, instead of a single rod, and is surmounted by a *spirit-level*, capable of being raised or lowered by screws at either end, so that the bulb of air in the tube can be made to occupy the middle of the tube, and indicate a perfect level. Above or below the spirit-level, a telescope is placed, having its axis parallel to that of the tube, and an observer is thereby enabled to read off more accurately the divisions of the second staff.

262. When the line to be measured, as from A to G, is upon ground which falls and rises between its ends, then if B, C, D, E, F be the points where there is a change from fall to rise, or rise to fall, and Ag, Gg, be the lines whose length we desire to know, then AB, BC, CD, DE, EF, FG, and the perpendiculars Bb, Cc, Dd,



Ee, *Ff*, *Gg*, can be successively found as *AB* and *DA* were in (260): and we have

$$Ab = \sqrt{AB^2 - Bb^2},\ bc = \sqrt{BC^2 - (Bb - Cc)^2},\ \&c.,$$

and thus *Ag* is obtained.

If, however, the inclination is known in *degrees*, and the horizontal distance is required, we may, for every continuous slope, by the subjoined Table tell how much to throw off from the measured distance on the slope, to obtain the true *horizontal* distance.

The table is calculated for intervals of 30′, or half a degree; and enables us to find the comparative values of

Angle.	Reduction.	Angle.	Reduction.	Angle.	Reduction.
3°	·15	9°	1·23	15°	3·41
3½	·19	9½	1·37	15½	3·64
4	·24	10	1·53	16	3·87
4½	·31	10½	1·67	16½	4·12
5	·38	11	1·84	17	4·37
5½	·46	11½	2·01	17½	4·63
6	·55	12	2·19	18	4·89
6½	·64	12½	2·37	18½	5·17
7	·75	13	2·56	19	5·45
7½	·86	13½	2·76	19½	5·74
8	·97	14	2·97	20	6·03
8½	1·10	14½	3·19	20½	6·33

the sloping and horizontal lines *AB* and *Ab*, and indicates the actual reduction to be made in the measured line *AB*, to produce *Ab*. The reduction is given in tenths and hundredths of the unit: and the range of inclination is from 3° to 20½°.

Thus, for an angle of 9° the reduction is 1·23 links in 100, or the true value of 1 chain of slope, in the horizontal line, is (100 − 1·23), or 98·77, links, or ·9877 of a chain.

Ex. A line was measured 17·55 chains on ground having a continuous rise of 9°; required the horizontal length of the line.

The whole reduction

$$= 17 \cdot 55 \times 1 \cdot 23 \text{ links},$$
$$= 21 \cdot 5865 \text{ links};$$

and this subtracted from 17·55 chains leaves the horizontal length

$$= 17 \cdot 334135 \text{ chains};$$

or, at once, the horizontal length

$$= 17 \cdot 55 \times \cdot 9877 \text{ chains},$$
$$= 17 \cdot 334135 \text{ chains}.$$

It is usual to neglect all quantities below links, unless amounting to half a link.

263. NOTE.—The student who understands a little *Trigonometry* will know that Ab can be obtained from AB, by measuring the $\angle BAb$, and looking in a Table of logarithms for what is termed the *cosine* of BAb. This *cosine* is the value of the fraction $\frac{Ab}{AB}$.

For example, if BAb were 18°, its *cosine*

$$= \frac{\sqrt{10+2\sqrt{5}}}{4} = \frac{\sqrt{14 \cdot 472}}{4} = \frac{3 \cdot 804\ldots}{4} = \cdot 951\ldots;$$

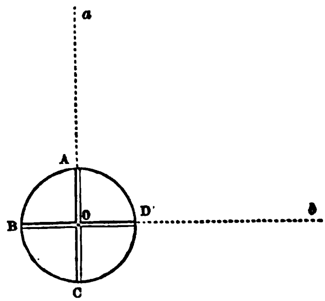
and if $AB = 2$, then $Ab = 1 \cdot 902\ldots$

We have observed that it is often needful to set off a straight line *at right angles* to another straight line upon the surface of the ground, as in measuring the perpendicular altitudes of triangles, or the lengths of offsets.

For measuring such perpendiculars, when very short, we may with a tape or offset-staff guess, with sufficient accuracy by the eye, the required perpendicular position of the lines; but for long lines, where any error would be more serious, we employ an instrument called

THE CROSS-STAFF.

264. The Cross-Staff, in its simplest form, consists of a circular board, as *ABCD*, about two inches thick, mounted on a pole about $5\frac{1}{2}$ feet high, and indented on its upper surface by two grooves, *AC*, *BD*, passing through the centre, *O*, at right angles to each other.



The use of it is to set off straight lines at right angles to other straight lines, as mentioned above.

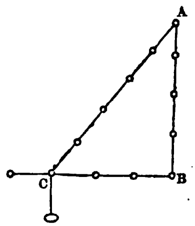
Thus, for example, (*see fig. p.* 276) the surveyor, in measuring *BC*, wishes to determine the point *D*, where the perpendicular from *A* meets *BC*. When he comes to the point, which he *supposes* to be nearly right, he there fixes his Cross-Staff in the ground, and turns it round, until one of the grooves is exactly over the line *BC*. He then looks along the other groove for *A*; and, without turning the board, or deviating from the line *BC*, he moves the whole instrument to the right, or to the left, until he sees *A* through the second groove. The foot of the staff is then on the required point *D*.

Similarly, in (255), the points *a*, *b*, *c*,... from which the offsets had to be measured, would be determined as above, unless the offsets from those points were so short, that it might be considered sufficiently correct to fix their position solely by the eye.

The accuracy of the instrument may thus be tested. Let it be so placed, that through *CA*, *BD*, two objects *a*, *b*, may be seen respectively: turn the cross half-round, and if *a*, *b*, are now seen through *BD*, *AC*, respectively, the instrument is correct.

265. The *Chain* itself also can be used sometimes conveniently for constructing a right angle. For from (43, Part I.) we know, that the square of the hypothenuse is equal to the sum of the squares of the other sides in a right-angled triangle.

Take, then, 12 links of the chain, and having laid down 4 of them in the direction AB of the line to which you wish to draw a perpendicular, so that the ends cannot move, divide the remaining 8 links into two lengths of 5 and 3 respectively, and pull them tight: the three lengths will form a right angled triangle ABC, where CB will be at right angles to AB, because $5^2=3^2+4^2$, or $25=9+16$.



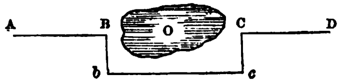
Since 3, 4, and 5, links are very short lines on the ground, the above method of setting off a right angle, although theoretically true, requires some modification in practice, because a short line of 3, or 4, links could not be *continued* without risk of serious error. But, remembering the numbers 3, 4, 5, we may adopt *any multiple* of those numbers at pleasure, as 30, 40, 50. Thus, if AB be measured 40 links, and 80 links of the chain be made to measure AC and BC, viz. $AC=50$, and $BC=30$, then $\angle ABC$ is a right angle, since

$$30^2+40^2=2500=50^2.$$

We now proceed to consider the difficulties arising from the intervention of *obstacles*, or any other cause of *inaccessibility*.

266. *To continue the measurement of a straight line with the chain, when some obstacle, as a pond, or building, or river, intervenes.*

Let AB be the straight line which the Surveyor is measuring, and B the point where the obstacle O interrupts the continuance of the measurement.

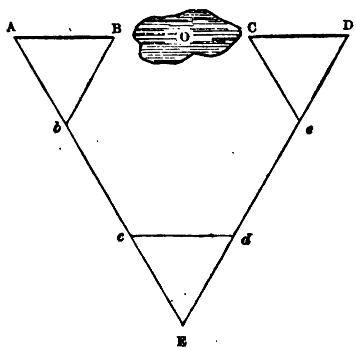


1st. At B set off by the *Cross-Staff*, or with the chain alone according to

(266), the line Bb at right angles to AB, and of such a length that it will not only clear the obstacle O, but terminate in a point b, where a parallel to AB can be readily set off and measured. Measure Bb; at b set off, at right angles to Bb, the line bc, of such a length that it will clear O, and allow a line to be set off and measured at right angles to it. Measure bc; and at c set off cC at right angles to bc, and equal to Bb. Lastly, at C set off a line CD at right angles to cC. Then it is plain, that, if BC be supposed joined, $BbcC$ is a parallelogram, and BC required is equal to bc, which was measured. Also CD is in the same straight line with AB, and therefore the measurement can be proceeded with as if the obstacle had not intervened.

If the obstacle does not impede the surveyor's view beyond it, an assistant may set up two poles at C and D, in the straight line continuous with AB, and then it will not be necessary for him either to measure cC, or set off a right angle at C.

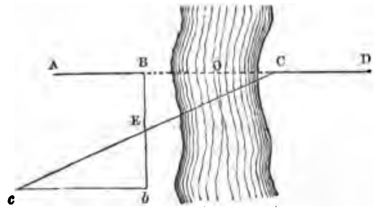
2ndly. The measurement of BC, and the line of direction CD, may both be found by the *Chain* alone, without the Cross-Staff, as follows:—



In the direction of the line measure the portion AB equal to any convenient length, say 50 links; and then set off the equilateral triangle ABb, by stretching a chain *double* the length of AB, in this case 100 links. Continue the measurement in the direction bcE, where E is a point, from which, as nearly as can be judged by the eye, a line to O would be perpendicular to the direction of AB produced. Measure bE, and take $Ec = AB = 50$ links; and, as before, form the sides Ed, dc, each $= 50$ links, then Ecd is equilateral. Continue the measurement in the

direction Ed, making $ED=AE$. In DE measure $De=50$ links, and form the sides DC, Ce, each $=50$ links. Then CD is in the same direction as AB, and A, D, E, are the angular points of an equilateral triangle; also, since AB, CD are equal to Ab, cE, $BC=bc$, which has been already measured.

3rdly. Let the obstacle be a river, or steep ravine, which prevents measurements being taken as in either of the former cases.



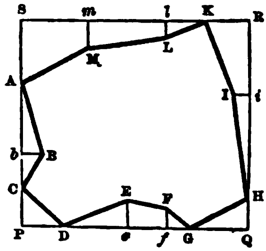
1st. An assistant on the other side of the river, or ravine, fixes a staff at C to range with A and B, and another at D in the same line. At B set off a line Bb at right angles to AB, and equal to any convenient *even* number of links of the chain; bisect it in E, and there fix a staff. At b set off bc at right angles to Bb; measure bc of such a convenient length that the staff C on the other side of the river, and in a line with AB, is seen in the same line with E and c. Then bEc, BEC, are similar triangles, and $BC=bc$, as before.

N.B. BEb may be drawn at *any* angle to AB, provided there be means at hand for setting off bc at the *same* angle with BEb produced, so that the triangles may still be similar.

267. *To measure a wood, or a lake.*

As an example of difficulties met with in obtaining the area of a piece of land from the intervention of obstacles, we may take the case where it is required to measure an area of very irregular form, but bounded by lines nearly straight, and containing wood, or water, so that it cannot easily be traversed in every direction. It will then be best measured by observations taken from *without*.

Let $ABCD \ldots M$ be such a plot. At its outermost points let poles be placed that can be readily seen at some distance; and let four stations, P, Q, R, S, be chosen, so that four imaginary lines drawn through them at right angles may embrace the plot within the smallest rectangle, and pass through as many as possible of the points A, C, D, &c. The right angles at P, Q, R, S, will be determined by the *Cross-Staff;* and when the correct position of these points has been ascertained, let poles be placed to mark those positions. Then if the portions intercepted between the boundaries of the rectangle and the plot be measured, and the area so found be subtracted from the area of the rectangle, the remainder will evidently be the required area of the plot.



From every angular point in the boundary of the plot, which is not also in the boundary of $PQRS$, let a perpendicular be supposed to be drawn to the nearest side of the rectangle, and let a pole be placed at its intersection with that side; then the areas between these offsets and the boundaries of the plot and rectangle will be either triangles or trapeziums, which can be measured, as in (255).

Let the surveyor now commence at one of the angular points of the plot where it meets the rectangle $PQRS$, as D, and measure successively along the sides PQ, QR, &c., in the direction PQ; let him note the length measured, at every point where he meets with any of the poles placed as above described; and let each offset be measured, either with an offset-staff, or a second chain; in this case they will all be to the left of the surveyor. Let the registered calculations and observations be as follows; where all the lines have their length expressed in links; and therefore the areas will be expressed in square links which can at once be converted into acres.

220	695	to D
	475	
115	345	
From	$\odot A$	
310	890	to A
105	615	
56	245	
From	$\odot K$	
200	870	to K
90	520	
From	$\odot H$	
135	870	to H
	595	
80	525	
95	315	
From	$\odot D$	

Begin at D.

Taking the doubles of all the areas as in (255), we have

Twice DEe $= 315 \times 95 =$ 29925 sq. links.
" $EefF$ $= 175 \times 210 =$ 36750 "
" FfG $= 70 \times 80 =$ 5600 "
" GQH $= 275 \times 135 =$ 37125 "
" HIi $= 520 \times 90 =$ 46800 "
" $IiRK$ $= 290 \times 350 =$ 101500 "
" KLl $= 245 \times 56 =$ 13720 "
" $LlmM$ $= 161 \times 370 =$ 59570 "
" $MmSA$ $= 415 \times 275 =$ 114125 "
" ABC $= 475 \times 115 =$ 54625 "
" CPD $= 200 \times 220 =$ 44000 "

2)543740

$\therefore$ sum of outer areas = 271870 sq. links.

or 2·7187 acres.

Area of rectangle $PQRS$

$$= PQ \times PS = 1090 \times 985 \text{ sq. links}$$
$$= 10{\cdot}7365 \text{ acres}$$

Area of small triangles and trapeziums $\Big\} = 2{\cdot}7187$ „

$\therefore$ required area $= 8{\cdot}0178$ acres

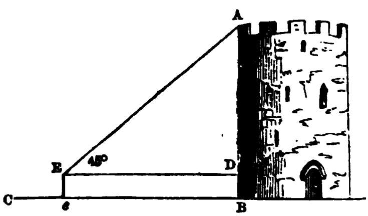
$$\begin{array}{r} 8{\cdot}0178 \text{ acres} \\ 4 \\ \hline {\cdot}0712 \text{ roods} \\ 40 \\ \hline 2{\cdot}8480 \text{ perches.} \end{array}$$

Or, area $= 8$a. 0r. 3p., nearly.

INACCESSIBLE HEIGHTS AND DISTANCES.

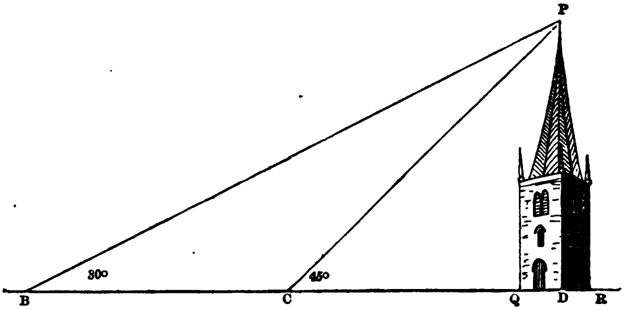
The measurement of these can hardly be said to be a part of Land-Surveying; but from the interest attaching to such problems, and the ease with which most ordinary questions of the kind can be worked, it has been thought worth a slight notice. Such measurement, by simple means, and without the aid of trigonometrical formulæ or calculations, depends upon a dexterous use of the particular angles 30°, 45°, 60°, 90°. Thus

268. If it be required to ascertain the height of a tower AB, to the base of which we can measure in the horizontal line BC; let the observer move backwards from the tower AB till the angle AED is found, by some instrument for that purpose, to be



45°; then $\angle EAD$ is also 45°, and $\therefore AD = ED = eB$, which can be measured. Adding Ee, or DB, the height of the observer's eye from the ground, we obtain the whole altitude AB.

Or, suppose the base of the object be not accessible, or the side opposite to the observer be sloping, as that of a tower surmounted by a steeple, or a pyramid, as PQR.



Let D be supposed to be the inaccessible point where a vertical line from P will meet the ground; DCB an horizontal line through D in any convenient direction; as before, let C be the point in this line where CP makes with CD an angle of 45°; then $PD = CD$. Let B be another point in the same horizontal line, such that the angle $PBD = 30°$; then, by (Prob. 2, p. 239), $CB = CD$, $\therefore PD = CD = CB$, which can be measured.

269. Tie, Check, or Proof, Lines.—To insure accuracy in the plotting and measurement of a small survey, made only with a measuring line of some sort and staves, it is usual to measure *upon the ground* a certain subsidiary line, which shall connect some two of the principal *measured*, and therefore *known*, lines of the survey; and then to make use of it, as a test, or check, in drawing the plan.

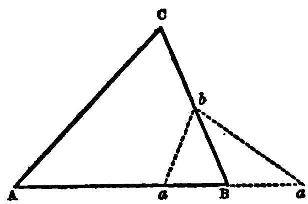
Such a line is commonly called a *Tie*, or *Check*, or *Proof*, *Line;* and it may be taken, so as to be situated either *within* or *without* the survey, according as it may be convenient to measure it on the ground.

Generally, it will be most advantageous to measure the *proof-line*, so that it shall form a *triangle* with the parts cut off by it from the two principal lines of the survey with which it is connected.

This may always be done *within* the survey, by making the *proof-line* intersect any two *contiguous* boundary lines; and *outside* the survey, by making it intersect the same two lines *produced*, if necessary.

These *check-lines* are not only in most cases especially useful as *correctors* in plotting, but are sometimes indispensable, where, from the nature of the ground, it would be either very difficult, or impossible, to measure diagonals and perpendiculars *within* the plot to be surveyed. Thus,

Ex. 1. Let the plot of ground to be planned and measured be in the form of a triangle, as ABC. The usual measurements being made, viz. the lengths AB, BC, AC, the ground may, of course, be plotted in the usual way *to a certain scale*. But it is possible there may be some error in the work; and to satisfy himself that there is none, the surveyor, instead of repeating the former measurements, fixes a staff, when he measures AB at some convenient point a, taking care to note down the distance aB. Similarly, in measuring BC, he leaves another staff at b, taking care to note down the distance Bb. Finally, he measures the '*proof-line*' ab. Then, having drawn his plan of ABC to scale, he lays down Ba, Bb, according to the same scale, and, if his plan be correct, he finds that, upon the scale being applied to ab, it agrees with his measured distance of a from b.



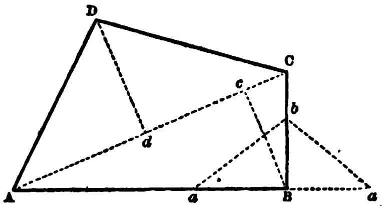
If the nature of the ground be such, that ab cannot conveniently be measured *within* the survey, the point a may be taken in AB *produced*, and ab measured *outside*.

Having thus obtained a correct plan of the ground drawn to scale, the area will be found by Art. (253).

Obs. If ab be used not as a *proof-line*, but simply as a *tie-line*, it is obvious that AC need not be measured at all.

Ex. 2. Let the plot of ground to be planned and measured be quadrilateral, as $ABCD$.

The usual measurements being made, viz. the sides *AB*, *BC*, *CD*, *DA*, and the diagonal *AC*, the ground may be plotted in the usual way to scale.



Also the perpendiculars *Dd*, *Bc*, upon *AC* being drawn, and calculated on the same scale, the area of *ABCD* is found by Art. (253).

Then *ab* is drawn as a *proof-line*, just as in the last Ex., either within or without *ABCD*, and serves at once to prove the correctness, or otherwise, of the survey.

It is further to be observed, that if, from the nature of the ground, it be difficult to measure *AC*, that measurement may be avoided by taking *ab* instead, as a *tie-line* either within or without the plot.

Thus, *proof-lines* are employed not only as *tests* of the correctness of the work of the surveyor; but also to enable him to plot and measure areas—such as those of woods, lakes, &c.—with certainty and ease, which otherwise could not be surveyed by means of simple instruments.

270. The content of irregular fields, farms, estates, parishes, or even whole counties, when correctly planned *to scale*, is sometimes found by a very ingenious and simple method, as follows:—

The plan being drawn upon paper, or drawing-board, of uniform thickness and texture, the portion whose area is required is cut out accurately along its boundaries with a sharp knife. Then from the same sort of paper, or drawing-board, a *square* is cut, which shall represent, according to the scale employed, *a known area*, such as an acre, or square chain, or a square mile,

&c. The two pieces of paper are then *weighed* in a very accurate balance, and the ratio of their weights will be that of the areas contained in them. So that, the area of the square being known to be an acre, or a square chain, or a square mile, as the case may be, the number of acres, &c. in the irregular plot is determined.

QUESTIONS AND EXERCISES L.

(1) What is the area in acres of a rectangular plot of ground, 128 yds. long, and $50\frac{3}{4}$ yds. wide?

Ans. 1a. 1r. $6\frac{3}{4}$p.

(2) State the advantage of taking the length of 22 yds. for the common *Chain*.

(3) When the sides of a rectangular plot are known in chains and links, how is the area obtained in acres, roods, &c.?

(4) Find the area, in acres, of a square whose side is 15 chains, 40 links. Ans. 23a. 2r. $34\frac{1}{2}$p.

(5) Shew how to find the area of a plot, of which an accurate plan has been obtained, without again going over the ground.

(6) Compute the area of a rhombus, whose base is 5 chains, 32 links, and perpendicular height is 3 chains, 7 links. Ans. 1a. 2r. $21\frac{1}{3}$p. nearly.

(7) Explain how the chain may be made use of for constructing a right angle.

(8) Give a brief description of the mode in which curvilinear fields are measured.

(9) Shew how, without using any offsets, a fair approximation may be made to the area of a field whose sides are not exactly straight lines.

(10) State the mode in which the observations taken in measuring a plot of ground are registered for future calculation; and write out an example of the method.

(11) When a piece of land, bounded by straight lines, is being measured, and it is not easy to traverse it,

shew how to lay down a correct plan, and thence to obtain its area.

(12) Make the largest right-angled triangle which can be constructed out of the links of two *Chains* fastened together; how many links are there to spare?

Ans. The sides will be 48, 64, and 80 links; and there will be 8 links to spare.

(13) In the triangle mentioned in the last question, what multiples are its sides of those of the triangle described in Art. (266)?

Ans. 16 times as large.

(14) In measuring a surface which differs considerably from the horizontal, what deviation must be made from the mode of measuring a level surface?

(15) A field is inclined to the horizon at an angle of $12\frac{1}{2}^\circ$ in the direction of its length; find from the Table in (263) the length of the projection, on the horizontal plane, of a line which on the slope measures 147·5 yards. Ans. 144·00425 yds.

(16) A road rises uniformly at the rate of 300 feet per mile; what is the difference between a mile measured on the slope, and the projection of that length on the horizontal plane? And what is the angle made by the road with the horizon?

(1) Ans. 2·85 yds.; (2) Ans. Rather more than 3°.

(17) A road, a mile long, makes an angle of 5° with the horizon; what must be the length of a canal which runs parallel to the road throughout the mile?

Ans. 1753·312 yds.

(18) How would the rise per mile be estimated in yards from the data in Ex. (17), by simple arithmetical calculation, without any levelling?

Ans. It is equal to $(1760)^2 - (1753\cdot312)^2$, in yds.

(19) On a piece of land having a uniform inclination of $5\frac{1}{2}^\circ$ to the horizon, a line was measured 11·73 chains long, in the direction of the inclination; required the distance, estimated longitudinally, over which the surveyor has passed. Ans. 11·676 chains.

(20) A hill is inclined to the horizon at an angle of 15° towards a river 2·5 chains broad; from its top a line is measured of 12·35 chains, in the direction of the slope; the hill rising on the other side of the stream, from its edge, is inclined at an angle of 10°, and measures to its ridge 7·5 chains; find the horizontal distance between the tops of the hills. Ans. 21·8144 chains.

(21) Describe the *Cross-Staff* and its use; and state the points upon which its accuracy depends.

(22) Find the areas of two pieces of land from the following notes; the measurement of the former was taken in yards, the latter in links:

	70	
6	56	
24	45	
17	23	
	0	

	93	
	65	32
28	36	16
	14	9
25	0	

(1) Ans. 1003½ yards.

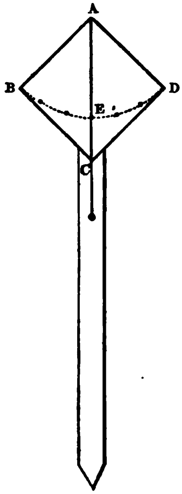
(2) Ans. 1r. 5·9p. nearly.

MEASURING INSTRUMENTS.

Certain '*Measuring Instruments*' have been already described, both as to their construction and use; *viz.* the *Tape*, and *Chain*, for measuring lengths or distances; the *Protractor*, for measuring angles; the *Offset-Staff*, for measuring short offsets; the *Cross-Staff*, for setting out lines *at right-angles* in the field; the *Level*, for finding the difference of level between any two given points, &c. But there are many other instruments of great value, when we come to actual work, and some of these we will proceed to describe. Especially we are required to give (which has not yet been done), such instruments as are commonly used for measuring *angles out of doors;* as, for instance, the angle contained by two sides of a field, considered as straight lines, and meeting at a point which is accessible; or, again, the angle in a vertical plane subtended at the eye of the observer by a lofty tower, or other building.

THE QUADRANT.

271. A very simple instrument may be made, as follows, for measuring, *in a vertical plane*, some of the more simple angles, as 15°, 30°, 45°, 60°, 75°, and 90°. Let $ABCD$ be a square board of convenient size. Draw upon it the diagonal AC; and with centre A and radius AB describe the arc of a quadrant BED, cutting AC in E. Then $\angle BAE = \angle DAE = 45°$. At E put 45, to shew that BE is an arc of 45°. With centre B, and radius AB, as before, set off the arc of 60°, and mark the point 60. Then from D also set off an arc of 60°, and mark that point 30, because it will determine an arc of 30° *measured from B*. Bisect this latter arc, and mark the point of bisection 15. Also bisect the arc D 60, and mark the point 75. The whole quadrant is now divided into 6 equal arcs of 15°.



This square board is so fastened to a staff, about 6 feet long, with a sharp point to enter the ground, as to permit it to revolve in its own plane round a fixed axle at A; and from A a plumb-line is suspended, which serves for adjusting the vertical position both of the staff and of the board.

This is a rough instrument for measuring such angles, *out of doors*, as are before mentioned, and may be used effectively for certain limited purposes. When used, the face of the board $ABCD$ is not only made vertical by means of the plumb-line, but it is turned round until it is in the *same* vertical plane in which the two points lie whose angular distance is required, and then the staff is fixed firmly in the ground. The observer, then, having the objects *before* him, whose angular distance he is to measure, places his eye at B, or D, as the case may be, and, looking along the upper edge of the board, he turns it round A, until he sees one of the

objects in that edge produced. In this position he notes where the plumb-line intersects the arc of the quadrant. He then brings the same edge of the board to the direction of the second object, and notes again the intersection of the plumb-line with the quadrant. The *difference* of the two graduations thus noted is the measure of the angle required.

But since only a few graduations are marked on this simple instrument, and since the observer can mostly select his own position, he should endeavour so to place himself, that, when he takes the *first* observation, the plumb-line shall pass over an exact division of the quadrant.

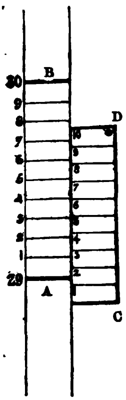
The instrument is especially useful in measuring the height of a lofty building, or tree, whose base is accessible. In this case a single observation only is needed. The observer takes up such a position that, when the instrument is rightly fixed, without moving the board at all, by placing his eye at B he sees the top of the building in BA produced. He then knows that the line BA, produced to meet the top of the building, makes an angle of 45° with the vertical; and therefore the height required is equal to the *horizontal* distance of B from the building (which is readily measured), with the difference of level between B and the foot of the building added or subtracted, as the case may be.

THE VERNIER.

272. The VERNIER, so called from the name of its inventor, is an instrument for measuring *very small* quantities on a graduated scale, either straight, as in the Barometer, or circular, as in the Theodolite, or Graphometer, or in many astronomical instruments.

Let AB be any convenient unit, as one inch, on the limb of an instrument, and divided into 10 equal parts. It is required to subdivide each of these 10 parts into 10 more, or the whole into 100 equal parts, without making 100 lines between A and B; for if they were made, no eye could count them.

For this purpose annex a small sliding instrument CD, called a vernier, equal to $\frac{9}{10}$ in., and divide it into 10 equal parts; then, since the whole CD is equal to $\frac{9}{10}$ in., each of its parts is equal to one-tenth of $\frac{9}{10}$ in. $= \frac{9}{100}$ in., or $\frac{1}{10}$ in. $- \frac{1}{100}$ in.; that is, each division of the vernier is less than one of the limb AB, by $\frac{1}{100}$ of an inch. Then, if the lowest points of AB and CD are made to coincide, the 2nd mark on the vernier is lower than the 2nd on the limb by $\frac{2}{100}$ in., or ·02 in.; the 6th on the vernier is lower than the 6th on the limb by $\frac{6}{100}$ in.; or ·06 in.; and so on.



The divisions of both the limb and the *vernier* are measured upwards.

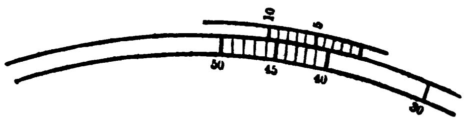
The mode of using the *vernier* will be seen from the accompanying diagram of the upper portion of a *Barometer.*

Let the upper end of the column of mercury stand somewhere between the 7th and 8th divisions of the 30th inch AB. Slide the vernier so that its highest point may come exactly opposite the head of the column. It is required to know by how many hundredths the upper end of the vernier is above the number 7 on the limb. Observe which division of the vernier corresponds with some one division of AB, so that they run into one horizontal line; let it be 6: then since each division of the vernier is *one*-hundredth shorter than the one in the instrument, the 7 on the vernier is *one*-hundredth below the division 5 on the limb nearest to it; 8 on the vernier is *two*-hundredths below 6 on the limb; and 10 is *four*-hundredths below 8 on the limb, or *six above* the 7; that is, the number 6, at the point where the divisions of the limb and vernier coincide, tells how many hundredths the top of the vernier is above the highest division reached by it on the limb. In this case, therefore, the reading of the height of the column will be 29 in. 7 tenths, 6 hundredths, or 29·76 in.

We see that the degree of accuracy to which the vernier measures is $\frac{1}{10}$ of $\frac{1}{10}$th of an inch, or $\frac{1}{100}$ in., because both limb and vernier were divided into 10 equal

parts. If they had each been divided into 20 equal parts, the accuracy would have been carried to $\frac{1}{20}$ of $\frac{1}{20}$th of an inch, or $\frac{1}{400}$ of an inch.

273. The limb may be *circular*, as in the next diagram; and here let the limb be divided into *degrees*, and the vernier, which is equal in length to 9 of these degrees, be subdivided into ten equal parts, as before: these will therefore enable us to measure tenths of a degree, or portions of 6 minutes. Let the extremity of the vernier fall, suppose, between the 45th and 46th degrees; and the



division marked 8 on the vernier coincide with some division on the limb, then the reading will be 45°, and 8 portions of 6 minutes, or 45° 48′.

Instead of taking the vernier one-tenth *less* than an inch, or than 10 degrees, we might take it one-tenth *more;* then, as before, the difference between the divisions on the limb and the vernier would be *one-hundredth;* but the divisions on the vernier would be numbered from the top. This is usually the case in the older barometers.

THE CIRCULAR PROTRACTOR.

274. It was mentioned in (236) that a more complete and accurate form of *Protractor* was used in actual practice than the one there described. It consists of a *complete* brass circle, crossed by a brass band. On each semicircumference is a vernier, described in (274), for reading *twice* any angle observed; (the importance of *two* verniers will be seen presently). The advantages of this instrument over the semicircular one seen in ordinary cases of instruments are as follows:—

I. In the semicircular instrument, [see fig. to Art. (233),] in order accurately to measure an angle already laid down on the paper, or to lay down an angle, it is necessary to make the straight edge *AB* coincide with the *middle* of the black stroke representing one of

the given *mathematical* lines which bound the angle, Now it is very difficult to do this with accuracy; but the desired coincidence is effected in the *circular* instrument, by placing the *Protractor over* the given line, so that two of the fine lines which mark the divisions on the limb, and which are always parts of a diametral line, may coincide with it; when, of course, the centre of the instrument will also be in that line.

II. If the centre of the instrument be not obtained with perfect accuracy, the error in any angle observed, and arising from this inaccurate position of the centre, called the *eccentricity* of the circle, is compensated for by measuring the vertical or opposite angle with the second vernier, which angle will of course give as much too large an arc, as the first did too small, or the converse; and the mean of the two observed angles is then taken.

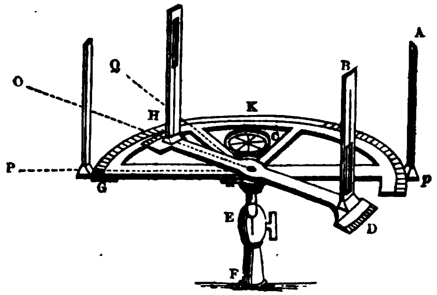
III. An angle also can be taken from the circumference instead of from the centre, as in [Prob. 16. (3) p. 252], by bringing the circumference of the *Protractor* over the angular point C, and observing the arc AB intercepted between the lines AC, BC, *half* of which will be the measure of the angle ACB.

THE GRAPHOMETER.

275. This instrument consists of a semicircle, or, which is much better, a complete circle of brass*, whose rim is divided into 180°, in the one case, and 360° in the other, with two diametral bands, one fixed, and the other moveable about the centre. At right angles to the extremities of each of these diametral bands is placed a *pinnule*, or *sight*. This consists of a thin, oblong, flat piece of brass, as represented at A and B, about 12 inches high, and having a slit pierced lengthways, and in the middle. One half of this slit is very narrow; the other is much broader, and is bisected lengthways by a wire, which, if continued, would also bisect the narrow part of the slit. At one end of each diametral band the narrow slit is up-

* The advantages of the complete circle over the semi-circle are well known to practical observers. And where great accuracy of measurement is of importance, the *semi-circular* instrument ought never to be used.

permost; at the other it is reversed. Each end of the moveable diametral band is furnished with a vernier for



reading off angles. The whole is attached centrally to a pivot, which works by a universal joint in an upright pillar resting upon a tripod. A compass and a spirit-level are attached, so that the diametral bands of the instrument can be placed in any required position, with respect to the points of the compass, and the plane of the circle be made horizontal.

The *graphometer* is used for taking angles, most commonly, either in an horizontal, or vertical, plane; but it may be turned in any direction, so as to bring it into the plane of *any* two or more objects whose bearings are required.

The observer looks through the narrow slit, and therefore has the larger opening of the opposite sight in the direction of the object observed. He finds the object through that latter opening, and brings the wire which bisects it into the same plane with the narrow slit close to the eye, so that the plane passing through the wires bisects the object.

Let E be the place of observation; O, P, and Q, objects in an horizontal plane, whose relative positions the observer wishes to ascertain.

Place the circle in the plane passing through O, P, and Q, and let the fixed diametral band be directed to P, so that to the observer at p the vertical plane, passing through the wires of A and its opposite sight, may bisect P. Direct the moveable diametral band DH, so

that the vertical plane through B and its opposite sight may bisect O. Then the number of degrees and the arc GH will measure the angular distance of O from P. The moveable diametral band is next directed to Q, and in like manner is obtained the angle PEQ; subtracting the previously obtained angle OEP, we obtain the difference OEQ.

As a correction of GEH, or OEP, HEp may be taken, and subtracted from 180°, giving a remainder GEH. If this differs from the value before obtained, the mean value will be found by taking half the sum of the two results.

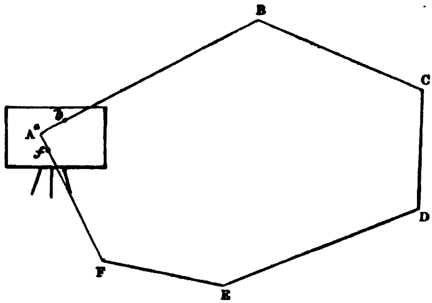
If the objects are in the same *vertical* plane, the instrument must be so placed by means of the joint at E, that the circular limb shall pass through them; and the sights will then have their wires horizontal.

In order that the diameter may be moved through a very small arc, so as to bring the wires exactly over the object, the end in contact with the semicircle is furnished with a clamp, whereby it is made to bite the limb; and its further motion, which was before produced solely by the hand, is then regulated by a screw with a milled head.

THE PLANE TABLE.

276. This instrument consists of a *plane* board of any convenient size, say 24 inches by 16, and on its upper face a piece of paper is affixed, upon which it is required to draw a plan of any plot of ground. The paper is securely fastened, either by broadheaded nails, or by pasting the edges. The board rests, as in the graphometer, on a tripod, in the upper part of which is a socket, wherein there drops a cylinder surmounted by a framework supporting the board. Between the cylinder and the framework is a universal joint, so that the board can be turned in any direction whatever. The horizontal position of the *Plane Table* is secured by the use of a spirit-level.

Suppose the proposed piece of ground to be of the form of the polygon $ABCDEF$: it may be correctly plotted by this instrument in one of the three following different ways.

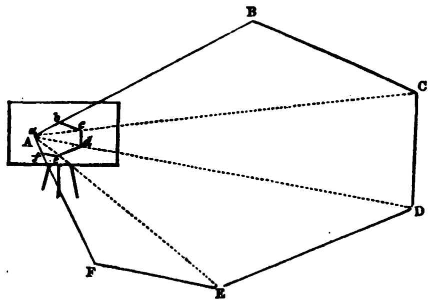


(1) Place the instrument at one of the angles, *A*: and after having secured the table being horizontal, make a mark on the paper exactly above the point *A*. This is done by means of a spirit-level and a plumb-line. Call the point so marked *a*.

At *A, B, C, D, E, F, G,* flag-staffs must be placed vertically.

Now at *a* place one extremity of one of the wires of the moveable sight, described in the graphometer, and with the eye at *a* make the staff at *B* to be just seen in the wire of the opposite sight; then draw through the wires in the direction *AB* an indefinite line upon the paper. Without moving the plane table, direct the eye in like manner to *F*, and draw an indefinite line in that direction: we now obtain the ∠*BAF*, or, on the paper, *baf*. Next send an assistant to measure successively *AB* and *AF* with a chain: set off *ab* and *af* upon the indefinite lines before drawn, representing *AB* and *AF*, according to some convenient *scale*. Then transfer the instrument to *B*, placing the point *b* of the partially drawn plan vertically over *B*. Direct the sight to *A*, making *ba* coincide in direction with *BA*; turn the sight to *C*, and thus obtain the ∠*ABC*. Draw the indefinite line *bc*, measure *BC*, and then lay down *bc* according to the scale agreed upon, and mark *c* on the plan. Proceed in like manner to obtain the other angles at *C, D, E, F,* and the lengths *CD, DE, EF*, which give *cd, de, ef*, on the plan: the whole outline is then plotted on the paper. It is not necessary to carry the instrument to *F*, as *af* was already traced, when the observer was at *A*.

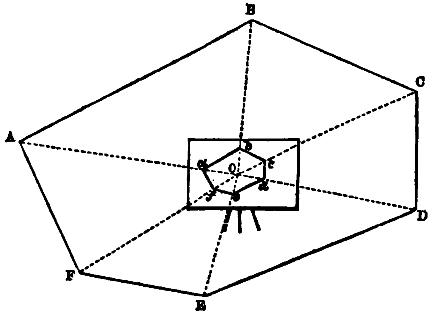
(2) If the staves at B, C, D, E, F, can all be readily observed from A, the process will be very much shortened.



Plant the table at A, as before; and mark the point a exactly above A. Without moving the table, direct the sight successively to B, C, D, E, F, and draw indefinite lines ab, ac, ad, ae, af. Let one or more assistants measure the lines AB, AC, AD, AE, AF, and lay down upon the indefinite lines ab, ac, &c.... distances representing AB, AC, &c. according to some convenient *scale:* we thus obtain the points b, c, d, e, f. Join these points, and a plan of the polygon is obtained, as before. This method has the great recommendation that the task of planting the table horizontally has to be performed but once.

(3) If all the angular points cannot be observed from any one of them as above, it often happens that they may be so observed from some central point, O, as in the next diagram. Then, as before, we direct successively the sight to the angular points of the polygon; trace the lines Oa, Ob, Oc,...upon the plan; measure OA, OB, OC...; and lay down upon the paper, Oa, Ob, Oc...according to *scale;* hence we determine the angular points, a, b, c...and by joining these points, we obtain $abcdef$, a plan of the proposed polygon, as in (1) and (2).

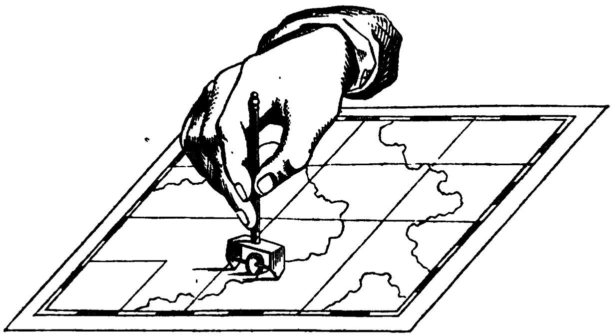
NOTE.—If the surface to be planned be of greater length than can be conveniently mapped upon a single sheet of paper of the size of the instrument, the board may be furnished with rollers on two of its opposite sides,



upon which a sheet of the required length is wrapped; and as a portion is completed, and wound round one roller, fresh paper may be unrolled from the other.

THE OPISOMETER*.

277. This simple instrument measures the length of any crooked lines, as roads, rivers, fences, walls, &c. on any map, or plan, which is *drawn to a scale*, without requiring any arithmetical calculation.



The principle of the *Opisometer* is, that, after having been applied to any line, it retraces or measures *backwards* precisely the same length *on the scale* with which the line is to be compared. It consists of a milled wheel with a steel screw for its axis, mounted on a convenient

* Made and sold by Messrs Elliott, Brothers, 30, Strand, London.

handle. To measure the length of a line, as the distance between two towns by the road traced upon a map, turn the milled wheel up to one end of the screw until it stops; and then place the instrument on the map, in an upright position, as represented in the diagram, the wheel resting upon one extremity of the line to be measured; then run the wheel along the road, following every bend as closely as possible. Care must be taken to keep the wheel in contact with the paper, but the pressure need not be such as to injure the map. When the wheel has arrived at the other extremity of the line, lift the instrument carefully from the paper, and carry it to the zero end of the *scale;* run the wheel *backwards* along the scale, until it stops at the same end of the screw from which the measurement began; the division of the scale, at which the wheel stops, shews the length of the line measured on the map. Should the *scale* be shorter than the line measured, when the wheel arrives at the end, carry it to the zero mark again as often as may be necessary, counting the number of repetitions.

The accuracy of the result given by the *Opisometer* is unaffected by the dimensions of the instrument itself, and depends entirely on the care with which it is used. The chief point is to see that the handle of the instrument is perpendicular to the surface at the beginning and end of each step of the measurement.

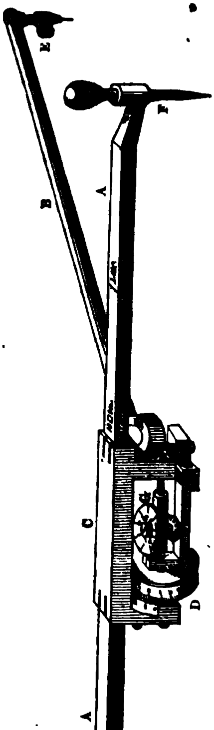
AMSLER'S PLANIMETER*.

278. This singularly beautiful instrument was lately invented by Professor Amsler, of Schaffhausen, by means of which the *area* of any portion of a map, or plan, *drawn to scale,* is readily and accurately measured, however irregular the boundaries may be.

The PLANIMETER, when ready for use, as in the annexed diagram, rests upon three points D, E, F; these are respectively, 1st a point of the circumference of the divided wheel D; 2ndly, a point of the tracer F, at the end of the arm A; 3rdly, a point E, at the end of the other arm B, which is kept *fixed* during the time of

* This instrument is to be had only from Messrs Elliott, Brothers, 30, Strand, London, and the price is £3 13*s*. 6*d*.

operation. To calculate contents, or areas, *in square inches*, set the slide *A* to 10 in. (as here shewn), which means that the result multiplied by 10, gives the content, or area, *in square inches*. To obtain this result, place the point *E*, at a convenient distance from the figure to be measured, so that the tracer *F* may traverse the entire periphery of the figure. But if the figure is too large to allow this, it can be subdivided by drawing straight lines through it, and the contents, or areas, of the several parts computed separately, and added together. Then place



the point of the tracer on any convenient starting point in the periphery. When the instrument is thus adjusted, read off the division on the horizontal disc *G*; also that on the perpendicular wheel and vernier, *H*. Suppose that the horizontal disc gives 3, and the vertical wheel gives 905, namely 90 on the wheel and 5 on the vernier; this reading must be put down thus, 3·905. Then carry the tracing point round the figure in the direction of the hands of a watch; and when the whole circuit has been made, observe the readings again. Suppose them 5·763; then subtract the former reading from the latter; the result will be 1·858; multiply this by 10, and you will get the content of the figure in square inches, namely 18·58 square inches. Notice must be taken whether the disc *G* has made an entire circuit; if so, 10 must be *added*, for every revolution, to the whole number. If the disc, in the above case, had gone once round, the second reading would have been 15·763. If twice round, 25·763; and so on. If the result is required in *acres*, multiply the above result by the number of acres in the square inch, *according to the scale* used in drawing the plan.

The other divisions on the arm *A*, may be easily applied to any scale, by finding the result of one square acre or chain, and using that number as the coefficient to give the content of the surface required.

The *proof* of the principle of this instrument, which can only be understood by advanced students, will be found in a Note at the end of the book.

THE THEODOLITE.

279. The *Theodolite* is the most complete and efficient instrument used by Surveyors. It measures, with great accuracy, the angular distance between any two visible objects, either in the horizontal, or in the vertical, plane; and it is not necessary, that, in the former case, the objects themselves be in the same horizontal plane, or in the latter case, that they be in the same vertical plane.

But the *Theodolite*, in its best form, is so complicated, that a minute description here of its various parts would serve no good purpose. The only way really to understand it, is to see and handle the instrument itself.

CURVED SURFACES AND SOLIDS.

280. Before proceeding to consider the mensuration of *Solids* generally, we may call attention to two cases of *curved surfaces*, where the surface can be *unwrapped*, so as to form a *plane* surface. Its area may then be obtained by methods already given.

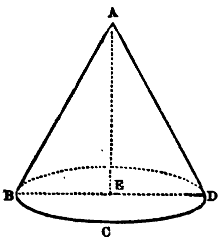
DEF. *A Right Cone* is a solid which may be supposed to be generated by the revolution of a right-angled triangle about one of its shorter sides as an axis, the conical surface being generated by the hypothenuse: the base of the cone is a circle.

DEF. *A Right cylinder* is a solid which may be supposed to be generated by the revolution of a rectangle

about one of its sides as an axis, the opposite side of the rectangle describing the cylindrical surface: the extremities of the cylinder are two equal parallel circles.

Now the curved surface of a cone, when slit down in a straight line from the vertex of the cone to the base, and unwrapped, becomes the sector of a circle; and that of a cylinder, slit down by a straight line at right angles to the base of the cylinder, and unwrapped, becomes a rectangle. Therefore we can measure the surface of a *right cone*, or of a *right cylinder*, by a very simple process.

281. *To measure the curved surface of a right* cone.



Let $ABCD$ be the *cone*, AE its altitude, and BE the radius of its base, which is a circle; then the area of its curved surface is equal to that of a *sector*, whose radius is AB, the slant side, and whose arc is the circumference of the circle $BCDB$.

Let BD, the diameter of the base, be measured; then the circumference of the base $= \pi \times BD$, (237).

And, therefore, the curved *surface* of the cone, which is equal to the area of the above-mentioned sector, is equal to half the product of the slant side and the circumference $BCDB$, by (241),

$$= \frac{AB \times \pi \times BD}{2}.$$

Exs. If $AB = 4$ in., and $BD = 3$ in.,

then the curved surface $= \pi \times 6$ square inches,

$= 3{\cdot}1416 \times 6$ sq. in. $= 18{\cdot}8496$ sq. in.

If AE and BE are known, AB may be found without further measurement. For since ABE is a right-angled triangle,

$$AB^2 = AE^2 + BE^2.$$

Thus, if $AE = 8$, and $BE = 6$, then

$$AB^2 = 64 + 36 = 100, \therefore AB = 10.$$

And the curved surface will be equal to

$$\frac{\pi \times 10 \times 12}{2} = 60 \times 3{\cdot}1416,$$

$$= 188{\cdot}496.$$

282. *To measure the curved surface of a* frustum *of a right cone.*

If the upper part of the cone were cut off by a plane parallel to the base, the lower part would be called a *frustum*, and the surface of the *frustum* would evidently be found by subtracting the surface of the small cone so cut off from that of the complete cone.

Suppose the section be made through the *middle* of AB, or AD. Then, the curved surface of the small cone

$$= \frac{1}{2}\left(\pi \times \frac{1}{2} AB \times \frac{1}{2} BD\right),$$

$$= \frac{1}{4} \times \frac{\pi \times AB \times BD}{2}.$$

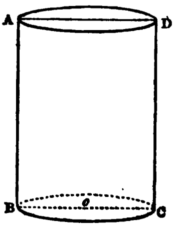
Subtracting this from the whole surface, we have area of the curved surface of the *frustum*

$$= \frac{3}{4} \times \frac{\pi \times AB \times BD}{2},$$

$$= \frac{3}{4} \text{ of the surface of the complete cone.}$$

A method of finding the surface of a *frustum*, independently of that of the complete cone, will be given in a future article.

283. *To measure the surface of a right* cylinder.



Let $ABCD$ be the *cylinder*, of which the height AB, and the diameter of the base, BC, are known.

Then the area of the curved surface is equal to that of a *rectangle*, whose adjacent sides are respectively AB, and the circumference of the base. This circumference $= \pi \times BC$; therefore the area of the *curved surface*

$$= \pi \times AB \times BC.$$

Exs. If the height of the cylinder be 10 in., and the diameter of the base be 4 in.; then, the area of the *curved* surface

$$= \pi \times 10 \times 4 \text{ sq. in.},$$
$$= \pi \times 40 \text{ sq. in.},$$
$$= 3{\cdot}1416 \times 40 \text{ sq. in.},$$
$$= 125{\cdot}664 \text{ sq. inches}.$$

In practice, we shall more often have the *circumference*, than the *radius*, of the base given; and if it be not given, it is readily measured. We have then nothing to do with π, since

Curved surf. = *Circumf.* × *height.*

Thus, if the circumference of a cylinder be $10\frac{1}{2}$ feet, and its height 5 feet,

Curved surf. $= 10\frac{1}{2} \times 5 = 52\frac{1}{2}$ sq. ft.

EXERCISES M.

[The value of π has been here taken as $\frac{22}{7}$.]

(1) A cylinder has a base whose radius is 1·75 ft.; and the height is 5·26 ft.; find the whole surface, including the two ends. Ans. 77·11 sq. in.

(2) The perpendicular height of a conical tent is 9 ft., and the radius of its base is $4\frac{1}{2}$ ft.; find the area of the canvas. Ans. 142·305 sq. ft.

(3) The radius of the base of a conical tent is $6\frac{1}{2}$ ft., and the length of the slant side is $9\frac{3}{4}$ ft.; find the length of canvas, $\frac{3}{4}$ yd. wide, required to make the tent.

Ans. $29\frac{32}{63}$ yds.

(4) The vertical angle of a cone is 60°, and the perpendicular height is 15 ft.; find the whole surface of the cone, including the base. Ans. 707 sq. feet, nearly.

(5) The curved surface of a cone is 132 sq. ft., and the radius of its base $3\frac{1}{2}$ ft.; find the length of its slant side, and the perpendicular height.

(1) Ans. 12 ft. (2) Ans. 11·03 ft.

(6) The inner surface of a cylindrical *pipe* is 56 sq. yds., and its length is 21 yds.; find the radius and area of an internal section of it perpendicular to its length. (1) Ans. $1\frac{3}{11}$ ft. (2) Ans. $5\frac{1}{11}$ sq. ft.

(7) Find the cost of plastering the walls of a cylindrical shaft, of which the height is 18 ft., and the base contains 616 sq. ft., at $4\frac{1}{2}$*d.* per square yard.

Ans. £3. 6*s.*

(8) An upright circular cup is $3\frac{1}{2}$ in. deep, and 2 in. in diameter, in the inside. The material of which it is made is $\frac{1}{8}$th of an inch thick. Find the number of square inches in its outer and inner surfaces.

(1) Ans. Inner surface = 44 sq. in.
(2) Ans. Outer ...... = $49\frac{1}{2}$ sq. in.

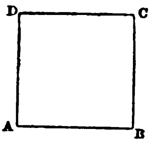
(9) To each end of an oblong room 30 ft. by 14 ft., and 12 ft. high, a semicircular recess is built of the same height; find the cost of painting the whole interior surface of the walls at 9*d.* per square yard, making no deduction for windows and fireplace. Ans. £5. 4*s.*

(10) A cone has two-thirds of its height cut off by a plane parallel to the base: compare the area of the curved surface of the small cone so cut off with that of the frustum remaining.

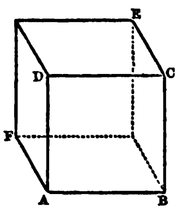
Ans. Surface of small cone = four-fifths of that of the frustum.

284. All *surfaces* of solids can be converted into actually equivalent, or approximately equivalent, *plane* surfaces; but the cone and cylinder only can be measured by the simple processes just explained, unless we add the *prism* and *pyramid*, of which we shall see that the *cone*, and *cylinder*, are particular cases. And any such will be best discussed, when we are treating of the measurement of the *volumes* of those solids.

285. We proceed then to investigate the general principles of measurement of *Solids;* and shall shew how to measure the *content*, or *volume*, of a *Solid*, that is, a body of *three* dimensions, length, breadth, and thickness, in like manner as it was before shewn how to measure a *surface* of *two* dimensions, length and breadth only.



We have seen that if, upon any straight line, as AB, a square $ABCD$ be described, this figure is called the *square of* AB, and is expressed by AB^2.



In like manner, if upon AB the annexed figure $ABCDEF$ be described, having six plane rectangular sides, or faces, each equal to the square $ABCD$, $ABCDEF$ is called the *cube of* AB, and is expressed by AB^3. Also, if AB be taken to represent 1 inch, 1 foot, &c., this figure will represent 1 *cubic inch*, 1 *cubic foot*, &c.

We have now therefore a third kind of unit of measurement; so that, on the whole, we find there are *lineal*, or common inches, feet, &c.; *square*, or superficial, inches, feet, &c.; and *cubic*, or solid, inches, feet, &c. Thus AB is a *lineal* inch, or foot, or &c.; $ABCD$ is a *square* inch, or foot, or &c.; $ABCDEF$ is a *cubic* inch, or foot, or &c. With the first unit we measure *lengths*, *lines*, or *distances;* with the second unit we measure *areas*, or *surfaces;* with the third unit we measure *contents*, or *volumes*.

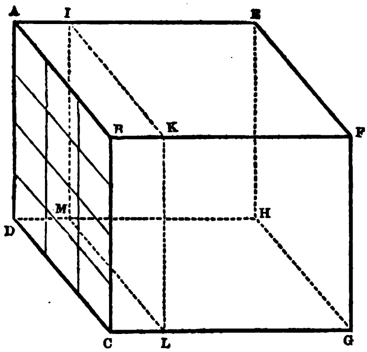
It has been shewn, that any rectangular plane surface

is measured by the product of its length and breadth (223), expressed in the same unit of lineal measure.

We have now to shew, in like manner, that

The volume, or content, of any solid bounded by six rectangular plane surfaces, is measured by the continued product of the number of units in the length, breadth, and thickness.

First, let all the dimensions consist of whole numbers, as 3, 4, 5, in.; and let $ABCGH$ be the solid where



$$AB=3, \ AD=4,$$
$$AE=5.$$

Let $AI=1$, and $IKLM$ be a section of the given solid by a plane parallel to $ABCD$.

Then, since $AE=5$, the whole solid may be divided by planes parallel to $IKLM$ into solid blocks, the base of each of which is $ABCD$, and the thickness 1 inch. Hence there will be five times as many cubic inches in the whole AG, as there are in the portion AL.

Now the area $ABCD=4\times3$ sq. inches (223). And if upon each of these square inches a cubic inch be placed, there will be 4×3 cubic inches in AL. Hence in the whole AG, or five times AL, there will be $5\times4\times3$ cubic inches, that is, the number of inches is denoted by the product obtained by multiplying the numbers representing the three dimensions, length, breadth, and thickness.

2ndly. Let the three dimensions be not expressed in *whole* numbers, as $5\frac{1}{4}$, $3\frac{1}{2}$, $1\frac{7}{8}$. And let each of these dimensions be divided into eighths (8 being the least common denominator of their fractional parts); then, the number of *eighths* in them will be 42, 28, 15, respectively.

Let three adjacent edges of the solid be divided into eighths, and through each of the points of division draw planes parallel to the several faces of the given solid; then the whole volume will evidently be divided into smaller *cubes*, each having for its edge one-*eighth* of the former lineal unit, and the *volume* of which will therefore be $\frac{1}{512}$th of the former solid unit. Also the *number* of these small cubes will, by the preceding case, be $42 \times 28 \times 15$; and therefore the measure of the *volume* of the given solid, in terms of the larger unit, will be

$$\frac{42 \times 28 \times 15}{512}, \text{ or } \frac{42}{8} \times \frac{28}{8} \times \frac{15}{8}, \text{ or } 5\tfrac{1}{4} \times 3\tfrac{1}{2} \times 1\tfrac{7}{8};$$

that is, *the product of the numbers representing the three edges*, as before, expressed in the same unit of lineal measure.

286. The solid bodies of which we shall investigate the measurement, both as to their *surface* and *volume*, are the *parallelopiped*, the *prism*, the *cylinder*, the *pyramid*, the *cone*, and the *sphere*.

From these may be deduced the measure of various other forms, as the *frustum* of a cone, the *barrel* or *cask*; the *pipe*, or *hollow cylinder*, the *wedge*; with many others which we meet with in practice. Indeed, solids of *any* form can be measured, either exactly, or approximately, by skilfully dividing them into several portions, each of which presents one of the above defined solid figures, and then taking the aggregate of the several parts.

287. *To measure the volume, and surface, of a* parallelopiped.

Def. *A parallelopiped* is a solid bounded by six plane surfaces, all parallelograms, and of which every opposite two are equal and parallel. When the parallelograms are *rectangles*, the solid is then called a *rectangular parallelopiped*.

An ordinary box, or book, is of this latter form.

The measure of the *volume* of a rectangular *parallelopiped* has been found in (285) to be the continued

product of any three of its edges which meet in one point. Thus, the *volume* of the rectangular *parallelopiped*, which has its three adjacent edges respectively equal to 3, 4, and 5 lineal feet, is equal to $3\times4\times5$ cub. ft.; i.e. 60 cub. ft.

Also, its *surface* = twice (length + breadth) × height
+ twice (length × breadth)
$= 2(4+5)\times3+2\times4\times5;$
$= (54+40)$ sq. ft. $= 94$ sq. ft.

DEF. The *cube* is a particular case of the *parallelopiped*, viz. when all the edges are *equal*.

The *volume* of a *cube* is therefore found by multiplying the number of units in the edge by itself twice. For example, if the edge were 4 feet, the *volume* would be $4\times4\times4$ cub. ft., or 64 cub. ft. The expression $4\times4\times4$ can be written 4^3, or, in words, the third *power* of 4, or 4 *cubed*. The number 4 is here said to be *cubed*, because the continued product of $4\times4\times4$ measures the *volume* of the *cube* whose edge is 4.

If the edge of the cube were 12 in., or the *cube* were a solid foot, then its *volume* would be $12\times12\times12$, or 1728, cubic inches. So also, if the edge were 3 ft., or 1 yd., then its *volume* $=3\times3\times3$, or 27, cubic feet. Hence we have the results given in the Table called 'Solid Measure'

1728 cubic inches = 1 cubic foot.

27 cubic feet = 1 cubic yard.

As the six planes forming the entire *surface* of the *cube* are all equal, and each is equal to the square described upon the edge of the cube; therefore the whole *surface* is equal to six times the square of the number measuring the edge. Thus, if the edge is 4 feet, the whole *surface* $=6\times4^2$ sq. ft. $=96$ sq. ft.

In the same article (285) it was shewn, that the *volume* of a rectangular *parallelopiped* is found in like manner, whether the numbers representing the lengths of the edges be *whole* or *fractional*. Hence the *volume* of a rectangular *parallelopiped*, the edges of which, meeting in one point, are 6 ft. 7 in., 3 ft. 9 in., and 2 ft. 6 in.,

$$=6\tfrac{7}{12}\times3\tfrac{3}{4}\times2\tfrac{1}{2}\text{ cub. ft.}=\frac{1975}{32}\text{ cub. ft.}=61\tfrac{23}{32}\text{ cub. ft.}$$

$$=61\text{ ft. }1242\text{ in.}$$

Ex. Let the dimensions of a rectangular parallelopiped be 10·5 inches, 2·05 ft., and 3·27 yds. Find the solid *content*, or *volume*, and also the *surface*.

Reducing the dimensions to the same unit, viz. feet,

$$\text{the } \textit{volume}=\left(\frac{10\cdot5}{12}\times2\cdot05\times3\cdot27\times3\right)\text{ cub. ft.,}$$

$$=\frac{70\cdot38675}{4}\text{ cub. ft.}=17\cdot5966875\text{ cub. ft.,}$$

$$=17\tfrac{3}{5}\text{ cub. ft., nearly.}$$

And the *surface*

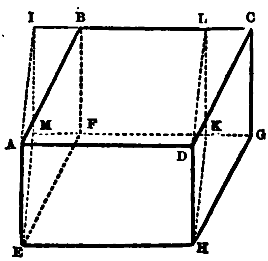
$$=2\times\{3\cdot27\times3+2\cdot05\}\times\frac{10\cdot5}{12}+2\times3\cdot27\times3\times2\cdot05,$$

$$=\frac{11\cdot86\times10\cdot5}{6}+9\cdot81\times4\cdot1,$$

$$=20\cdot755+40\cdot221=60\cdot976\text{ sq. ft.}$$

Next, let the *parallelopiped* be *oblique*, as $ABCDEFGH$, which has for its base the parallelogram $EFGH$, but not rectangular.

Through D draw the plane $DHKL$ perpendicular to AD or EH, and through A draw $AEMI$ also perpendicular to AD, meeting CB and GF produced in I and M. Then AK is a *rectangular parallelopiped.*



Also, since AI, AB are equal, and parallel to, DL and DC respectively; and since $AE=DH$; the volume of $ABIEFM$ is equal in all re-

spects to that of $DCLHGK$; and taking these equal volumes from the whole figure, we have the remainders equal, viz. the oblique parallelopiped EC is equal to the rectangular one AK.

But the *volume* of $AK = \text{base } HM \times AE,$

$= \text{base } EG \times AE;$

$\therefore$ *vol.* of oblique parallelopiped = *area of the base* × *ht.*

288. *To measure the volume, and surface, of a* prism.

DEF. *A Prism* is a solid bounded by plane surfaces, of which two (called the *ends* of the prism) are equal, similar, and parallel figures of any number of sides, and the rest parallelograms.

1st. Let the base of the prism be triangular.

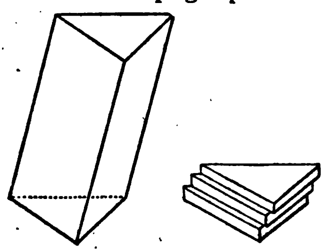
Suppose a plane to be made to pass through AE, CG, the opposite edges of the oblique parallelopiped of the last article; then that solid will be divided into two equal *prisms* with triangular bases; and the *volume* of each will therefore be equal to half the *volume* of the parallelopiped; that is, it will be equal to *the product of the height and the area of its triangular base.*

2ndly. Let the base be a polygon.

The *prism* may be divided into other *prisms*, having *triangles* for their bases by planes through pairs of edges; and, by the last case, the volume of each is equal to the product of the height and the area of its triangular base; therefore *the volume of the whole prism is equal to the product of the height and the area of the whole base.*

3rdly. Let the *prism* be oblique.

Take an upright *prism* of the same base and height; suppose it to be divided into any number of equal thin *prisms* by sections parallel to the base; and let them be placed, as in the figure, so that they may assume the form of an *oblique prism;* then, if the number of sections be indefinitely increased, and



therefore the thickness of each portion correspondingly diminished, the *volume* of all the small pieces will not differ appreciably from that of the oblique *prism*.

$\therefore$ *vol.* of the oblique prism

= sum of the vol^s. of all the small prisms,

= the vol. of the upright prism,

= *base* × *perp^r. height.**

Hence, in all cases, *the vol.* of a prism = *base* × *perp. ht.*

If the *surface* be required, since a *prism* is bounded by plane surfaces which are parallelograms, and by two triangular or polygonal ends, the whole *surface* is therefore obviously obtained by measuring all these plane surfaces according to the methods already given for measuring any plane surfaces whatever, and taking their sum.

Ex. Let the height of an upright *prism* upon a triangular base be 10 ft., and the sides of the base be 3, 6, 7, feet, respectively; find the *volume*, and *surface*, of the *prism*.

By (230) the area of each triangular end

$$= 8{\cdot}95 \text{ ft. nearly.}$$

$\therefore$ *volume* of prism $= 10 \times 8{\cdot}95 = 89{\cdot}5$ cub. ft.

Also, the three rectangular parallelograms bounding the *prism* have the same length, 10 ft., and the sum of all these

$$= 10 \times (3 + 6 + 7) = 160 \quad \text{sq. ft.}$$

Also, area of the ends = $\underline{17{\cdot}9}$ sq. ft.

$\therefore$ *whole surface* = $\underline{177{\cdot}9}$ sq. ft.

289. *To measure the volume, and convex surface, of a* cylinder.

* A pack of cards, pushed out of the perpendicular slightly, and equally from the lowest to the highest card, furnishes a good illustration of this case.

If the straight line joining the centres of the two ends of the cylinder be perpendicular to the base, the cylinder is a *right* cylinder; but if it makes any other angle, it is termed an *oblique cylinder.*

A *right cylinder* has already been defined in (281). The *surface* of a *right* cylinder has been found in (283), from its being capable of being unwrapped, and measured as a *plane* surface.

Also, since the *cylinder*, whether *right* or *oblique,* may be considered as a *prism,* upon a polygonal base with an indefinitely large number of sides, therefore, its *volume,* as in the case of a *prism,* is equal to *the product of its height and the area of the base.*

But if the cylinder be *oblique,* its convex *surface* may be found as follows:—

Suppose the surface be unwrapped, as in (283); then it will be found to be a portion of a circular *ring,* of which the two circular arcs are similar and equal, having their extremities joined by equal and parallel straight lines.

Now this plane area may be supposed to be divided into an indefinite number of small equal parallelograms, by lines parallel to the ends; and the area of each will be equal to its *length* $\times$ *its height.* Hence the whole area is equal to the product of the length, and the sum of all these small heights, i. e. the convex *surface* is equal to *the product of the length of the slant side of the cylinder, and the circumference of a section at right angles to the slant side.*

By another method the same result may be obtained:—

Let two sections of the *oblique* cylinder be made at right angles to the axis, and passing through the extreme *opposite* points in the circumferences of the two ends; the portions of the cylinder thus cut off will evidently be equal and similar in all respects; and if one of them be removed, and placed at the other end, neither the convex surface, nor the volume, of the cylinder will be altered. But we shall then have a *right* cylinder, whose height is equal to the slant side of the *oblique* cylinder, and base one of the sections before-mentioned; and its *sur-*

face = its height × circumference of base (283); therefore, for the *oblique* cylinder, *surface required* = *slant side* × *circumference of section at right angles to axis.*

Also, *volume* of oblique cylinder = *slant side* × *area of section at right angles to axis.*

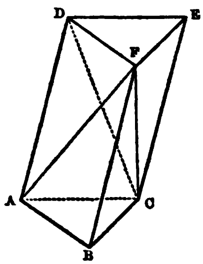
290. *To measure the volume, and surface, of a* pyramid.

Def. A *Pyramid* is a solid bounded by plane surfaces, of which one (called the *base* of the pyramid) is any rectilineal plane figure, and the rest are triangles converging to one point as a common vertex.

If the pyramid be placed with its vertex upwards, and a perpendicular from that vertex upon the base falls upon the *centre* of the base, or of any regular curve circumscribing the base, the figure is termed a *right* pyramid. If the perpendicular does not so fall, it is termed an *oblique* pyramid.

1st. When the base of the *pyramid* is a *triangle.*

Let $FABC$ be the *pyramid*, either right or oblique, having the triangle ABC for its base. Through the points A and C draw the lines AD, CE, equal and parallel to BF; and let planes pass through DA, FB, and FB, EC; and a third plane through the points D, E, F; we shall thus form a triangular *prism* $ABCDFE$, of which the proposed *pyramid* is a part, and having the same base and height as the *prism*.



From this *prism* take away the *pyramid* $FABC$; there remains a *pyramid* whose base is $ACED$, and vertex F.

Through D, F, C, draw a plane; the *pyramid* $FACDE$ will be thereby divided into two triangular *pyramids*, $FDAC$, $FDEC$, which are equal, because they

have the same height, namely, the perpendicular from F upon $ACED$, and equal bases, DAC, DEC.

Also, $FDEC$ can be considered as having its vertex at C, and base DFE, and therefore is equal to $FABC$, since the bases DFE, ABC, are equal, and the height is common; hence the three *pyramids* are equal; and therefore the *volume* of each of them is equal to one-third of the volume of the whole *prism*,

$$= \frac{1}{3} \textit{ base} \times \textit{perpendicular height} \ (288).$$

2ndly. When the base is a *parallelogram*.

Let F be the vertex, and $ACED$ the base of the *pyramid*, then it can be divided by a plane through FC, FD, into two *pyramids* on equal triangular bases ADC, EDC; and, by the previous case, the *volume* of each of these equal *pyramids* is one-third of the product of the base and perpendicular height; and therefore the *volume* of the whole *pyramid* on $ACED$

$$= \frac{1}{3} \textit{ area of base} \times \textit{ht.}$$

3rdly. When the base is a *polygon*.

The *pyramid* may be divided into a number of *pyramids* on triangular bases by planes through the common vertex, and the diagonals of the polygon; and since the volume of each of these

$$= \frac{1}{3} \text{ area of triangular base} \times \text{ht.};$$

$\therefore$ *volume* of the whole pyramid,

$$= \frac{1}{3} \text{ sum of all the triangular bases} \times \text{ht.},$$

$$= \frac{1}{3} \textit{ the whole base} \times \textit{ht.}$$

Hence, the *volume* of any pyramid $= \frac{1}{3}$ *base* $\times$ *perp. ht.*

Ex. A pyramid stands on a base which is an equilateral triangle. Given that a side of the triangle is 4 feet, and the height of the pyramid is 9 feet, find its *volume.*

$$Vol. = \frac{1}{3} \times 4\sqrt{3} \times 9 = 12\sqrt{3} \text{ cub. ft., (230).}$$

The lateral *surface* of a *pyramid* is obviously obtained by taking the sum of all the triangular surfaces which bound it.

291. *To measure the volume, and surface, of a* cone.

A *right* cone has already been defined in (281). If the line joining the vertex, and the centre of the base, is perpendicular to the base, the cone is a *right* cone; but if it makes any other angle with it, it is termed *oblique.*

The *surface* of a right cone has been found in (282), from its being capable of being unwrapped, and measured as a *plane* surface.

Since the *cone*, whether *right* or *oblique*, may be considered as a *pyramid* upon a polygonal base, with an indefinitely large number of sides, therefore the *volume*, as in the case of a *pyramid*, is equal to *one-third of the product of the area of the base and the height.*

Cor. Since the *volume* of the *cylinder* of the same base and height as the cone

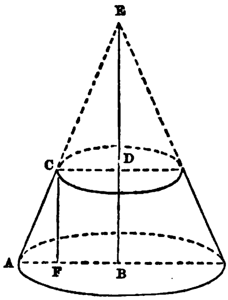
$$= \textit{area of the base} \times \textit{ht.}\ (289),$$

$$\therefore \textit{volume of a cone} = \frac{1}{3}\ \textit{circumscribing cylinder.}$$

Ex. The height of a conical tent is 12 ft., and the radius of its base is $5\frac{1}{2}$ ft., what is its volume?

$$\text{Volume} = \frac{1}{3} \times 12 \times \frac{22}{7} \times \left(\frac{11}{2}\right)^2 = 380\tfrac{2}{7} \text{ cub. ft.}$$

292. *To measure the volume, and surface, of a* frustum *of a right cone.*



Let B, and D, be the centres, and AB, CD, radii, of the circular ends of the *frustum;* and suppose the slant sides of the section through AB, and CD, to meet in E, the vertex of the complete cone, of which the *frustum* is a part. Let CF be drawn parallel to DB, the height of the *frustum*.

Then, since AFC, CDE are similar triangles,

$$DE : CD :: FC : AF,$$

$$:: BD : AB-CD;$$

$$\therefore\ DE=\frac{BD\times CD}{AB-CD}\ldots\ldots\ldots(1).$$

$$\text{Similarly } BE=\frac{BD\times AB}{AB-CD}\ldots\ldots\ldots(2).$$

$$\text{Also, } BD=\sqrt{AC^2-(AB-CD)^2}\ldots\ldots(3).$$

From these results the complete cone can be determined, and also the cone cut off. Their difference is the *frustum* required.

Or, the volume of the frustum may be measured approximately, *with a rough approximation*, by multiplying its height by the area of the section taken midway between the ends.

But the *correct* result obtained by the first method is given by the following Rule*, and is easily remembered: "*To the sum of the areas of the ends of the* frustum *add four times the area of the mid-section, multiply by the height, and take* one-sixth *of the result.*"

Ex. The radii of the ends of a *frustum* are 3, and 4 inches, respectively, and the length of its slant side is 3 inches; find the *volume* of the *frustum*.

$$\text{Here } AB-CD=1,\quad AC^2=9;$$

* It requires a little knowledge of *Algebra* to deduce this Rule from (1) and (2). It depends upon the fact, that AB^3-CD^3 divided by $AB-CD$ is equal to $AB^2+CD^2+AB\times CD$.

$\therefore\ BD=\sqrt{8}=2\sqrt{2},\ BE=8\sqrt{2},$ and $DE=6\sqrt{2}.$

By 1st method,

vol. of *complete* cone $=\dfrac{ht.\times area\ of\ base}{3}=\dfrac{22}{7}\times\dfrac{16}{3}\times 8\sqrt{2},$

$$=\frac{22}{21}\times 128\sqrt{2}.$$

vol. of *smaller* cone $=\dfrac{22}{7}\times\dfrac{9}{3}\times 6\sqrt{2}=\dfrac{22}{21}\times 54\sqrt{2}$;

$\therefore$ vol. of *frustum* $=\dfrac{22}{21}\times 74\sqrt{2}$ cub. in.

By 2nd method, since radius of mid-section $=\dfrac{3+4}{2}$, or $\dfrac{7}{2}$;

$\therefore$ vol. of *frustum* $=\dfrac{22}{7}\times\left(\dfrac{7}{2}\right)^2\times 2\sqrt{2},$

$$=\frac{22}{21}\times 73\tfrac{1}{2}\times\sqrt{2}\text{ cub. in.}$$

By Rule, vol. of *frustum*

$$=\frac{22}{7}\times 2\sqrt{2}\times\left\{\frac{4^2+3^2+4\times\left(\frac{7}{2}\right)^2}{6}\right\},$$

$$=\frac{22}{21}\times 74\sqrt{2}\text{ cub. in.}$$

Sometimes the *height* of the *frustum* is given, or it can be conveniently measured, as in the following

Ex. The height of the *frustum* is 6 inches, and the radii of the ends are 4, and $2\frac{1}{2}$, inches; then BE the height of the *complete* cone = 16 in., and DE, that of the smaller cone, = 10 in.; therefore,

By 1st method, vol. of *complete* cone $=\dfrac{16\times\pi\times 4^2}{3}$,

...... *smaller* ... $=\dfrac{10\times\pi\times(2\cdot 5)^2}{3}$;

$\therefore$ vol. of *frustum* $=\dfrac{\pi}{3}\times\{256-62\cdot 5\},$

$$=\frac{4257}{21}=202\tfrac{5}{7}\text{ cub. in.}$$

By 2nd method, since the radius of the mid-section

$$=\frac{4+2\cdot5}{2}=3\cdot25;$$

vol. of *frustum* $=6\times\pi\times(3\tfrac{1}{4})^2$,

$$=6\times\frac{22}{7}\times\frac{169}{16}=\frac{5577}{28}=199\tfrac{5}{28}\text{ cub. in.}$$

By the Rule, since the height of the *frustum* is 6 in.,

$$\text{vol. of } \textit{frustum}=6\times\pi\times\left\{\frac{4^2+(2\cdot5)^2+4\times(3\tfrac{1}{4})^2}{6}\right\},$$

$$=\frac{22}{7}\times\{16+6\cdot25+42\cdot25\},$$

$$=\frac{22}{7}\times64\cdot5=202\tfrac{5}{7}\text{ cub. in.},$$

as by the first method.

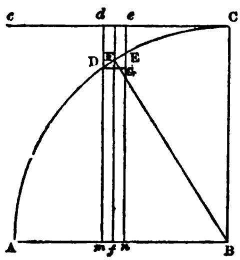
To find the curved *surface* of the frustum, suppose it split down in the line AC, and opened out until it becomes a *plane* surface. It then forms a portion of a *circular ring*, which has been measured in p. 244; and

$$\textit{surface}\text{ of frustum}=\pi\times(AB+CD)\times AC.$$

293. *To measure the surface of a sphere, when its radius, or diameter, is given.*

Def. *A sphere* is a solid generated by the revolution of a semicircle about its diameter.

Let ABC be a quadrant of a circle; and let it revolve completely round AB as an axis; it will describe a hemisphere.



Take a *very small* portion of the arc, DE; and draw Cc parallel to AB: through D and E draw dDm, and eEn, parallel to CB; and draw DG parallel to AB. Bisect DE in F, join BF, and draw Ff parallel to CB. Ff is the average distance of DE from mfn. Then, since

DE is very small, it may be taken as a straight line, and the surface generated by its revolution round AB, as that of a cylinder, with radius Ff, which will be $2\pi \times Ff \times DE$. (284). Also the surface of the small cylinder generated by the revolution of de round $mfn = 2\pi \times dm \times DG$. Now the triangles DEG, BFf, have each one right angle, and the angles EDG, BFf equal; therefore they are similar, and the sides about their equal angles proportional; hence

$$BF : Ff :: DE : DG,$$

$$\therefore\ BF \times DG = Ff \times DE, \text{ or } dm \times DG = Ff \times DE;$$

and therefore the area of the curved surface described by DE is equal to that of the *cylinder* described by de.

The same may be shewn to be true of all the successive very small portions of the arc of the quadrant ADC. Hence, the whole hemispherical surface generated by the revolution of ADC round AC

$= 2\pi \times dm \times$ (sum of all the small quantities like DG),

$= 2\pi \times dm \times AB = 2\pi \times (\text{rad.})^2$, since $dm = BC$;

and $\therefore$ *the surface of the whole sphere* $= 4\pi \times (\text{rad.})^2$.

Obs. The section of a sphere made by a plane through its *centre* is called a *great circle* of the sphere. Hence, the *surface of a sphere* = 4 *great circles of the sphere.* This is especially worth remembering.

Ex. If the diameter of the earth be 8,000 miles, what is its *surface* considered as a *sphere?*

$$\textit{Surface} = 4\pi \times (\text{rad.})^2 = 4 \times \frac{22}{7} \times 16{,}000{,}000 \text{ sq. miles},$$

$$= 201{,}142{,}855 \text{ sq. miles}.$$

294. *To measure the volume of a* sphere, *when its radius, or diameter, is given.*

A *sphere* may be considered as composed of a number of very small equal *cones*, having their common vertex at the centre, and the same altitude, viz. the radius of the *sphere*, and having as the base of each, a very small portion of the surface of the *sphere.*

Then the *vol.* of the *sphere*

$= $ sum of *vol^s*. of all these cones,

$= \frac{1}{3}$ ht. $\times$ sum of the areas of their bases,

$= \frac{1}{3}$ ht. $\times$ surface of the sphere,

$= \frac{1}{3}$ rad. $\times 4\pi \times (\text{rad.})^2$, (since height $=$ rad.),

$$= \frac{4\pi}{3} \times (\text{rad.})^3.$$

Cor. The *vol.* of the *cylinder* circumscribing the sphere, and of the same height as the diameter of the *sphere*,

$= $ ht. $\times$ base (289),

$= 2$ rad. $\times\ \pi \times (\text{rad.})^2$,

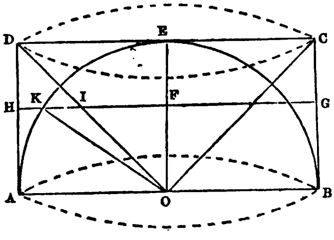
$= 2\pi \times (\text{rad.})^3$;

$\therefore$ *vol. of sphere* $= \frac{2}{3} \times 2\pi \times (\text{rad.})^3 = \frac{2}{3}$ *of circumscribing cylinder.*

Ex. The inner diameter of a foot-ball is 6 in.; how many cubic inches of air does it contain?

$$vol. = \frac{4\pi}{3} \times (\text{rad.})^3 = \frac{4\times 22}{3\times 7} \times 27 = \frac{22\times 36}{7} = 113\tfrac{1}{7}\ cub.\ in.$$

295. *Given that the volume of a cone is equal to one-third of the cylinder with the same base and height, prove that the volume of a sphere is two-thirds of the circumscribing cylinder.*



Let $ABCD$ be a cylinder,
AEB a hemisphere, O its centre;
ODC a cone, E centre of its base;
F *any* point in OE;
$GFIKH$ a straight line through F parallel to AOB.

Then $FH^2 = OK^2 = OF^2 + FK^2$,

$$= FI^2 + FK^2, \because OF : FI :: OE : ED$$
$$:: OE : OA$$
$$:: 1 : 1,$$

$$\therefore \ \pi \times HF^2 = \pi \times FI^2 + \pi \times FK^2,$$

i. e. *Circular section of cylinder*

$$= \textit{Circular section of cone} + \textit{circular section of } \tfrac{1}{2} \textit{ sphere},$$

and since F is *any* point in OE, $\therefore$ the same holds for *all* corresponding sections of the cylinder, cone, and hemisphere; and consequently for all corresponding *laminæ* of very small thickness. Hence

Whole Cylinder = whole cone + whole $\frac{1}{2}$ sphere;

But, by supposition, Cone = $\frac{1}{3}$ Cylinder,

$\therefore$ $\frac{1}{2}$ sphere = $\frac{2}{3}$ Cylinder.

Double the hemisphere, and also the Cylinder, then

vol. of Sphere = $\frac{2}{3}$ *Circumscribing Cylinder.*

296. Obs. We now see, that just as the area of a *circle* was proportional to the *square* of the radius, so the *volume* of a *sphere* is proportional to the *cube* of the radius. Hence the volumes of spheres are to one another as the *cubes* of their radii.

Thus, if two spheres have respectively radii of 9 in. and 5 in., the *vol.* of the larger : *vol.* of the smaller :: $9^3 : 5^3$

:: 729 : 125.

It may also be mentioned, that any *similar** solids, however irregular their shape may be, have also, as in the case of spheres, volumes proportional to the cubes of any two corresponding lines in them.

Thus, if AB, ab were corresponding slant sides or radii of two *similar cones;* or the heights, or radii, of two *similar cylinders;* and if it be known that

$$AB = \tfrac{4}{3}ab, \text{ or } AB : ab :: \tfrac{4}{3} : 1 :: 4 : 3,$$

* Def. *Similar* solids are such as have all their solid angles equal, each to each, and are bounded by the same number of *similar* plane surfaces.

Similar cones and cylinders are those which have their axes and the diameters of their bases *proportionals.*

then, *vol.* of larger solid : *vol.* of smaller $:: (AB)^3 : (ab)^3$,
$:: 4^3 : 3^3$,
$:: 64 : 27$.

Ex. How many spheres of 3 inches diameter can be placed in another of 12 inches diameter, supposing the small spheres made of plastic material, so as to fill the whole interior of the large sphere?

$$\frac{\text{Large Vol.}}{\text{Small Vol.}} = \frac{(12 \text{ in.})^3}{(3 \text{ in.})^3} = \frac{1728}{27} = \frac{64}{1};$$

i. e. the large sphere contains 64 small ones.

The same result may also be obtained thus. Since the large rad. = 4 times the small radius,

$\therefore$ Large vol. ; small vol. $:: 4^3 : 1 :: 64 : 1$, as before.

297. *To measure the solid matter of a pipe, or hollow cylinder.*

The pipe, or hollow cylinder, has been virtually included in the article on the cylinder, only that it is, as it were, one cylinder within another, and therefore its volume has not actually been measured.

Now the quantity of material employed in its construction may be found in two ways:

1st. By finding the volumes of both the outer and inner cylinders, and taking their difference.

2nd. By finding the surface of a cylinder which is a *mean* between the inner and outer cylindrical surfaces, i. e. whose radius is a mean of the outer and inner radii, and then multiplying the surface so obtained by the thickness of the material of which the pipe is composed.

Ex. How many cubic feet of iron are required to make a cylindrical chimney for a marine engine, which shall be 30 ft. high, have its inner radius 12 in., and thickness of metal three-fourths of an inch?

By the first method, the *volume* of the smaller cylinder $\Big\}= \pi \times 1^2 \times 30$ cub. ft.,

and (since $\frac{3}{4}$ in. $= \frac{1}{16}$ ft.), the *volume* of the larger cylinder $\Big\}= \pi \times (1\frac{1}{16})^2 \times 30$ cub. ft.;

$\therefore$ *volume* of iron = diff. of vol. of cylinders,

$$=\pi\times 30\left\{\left(\frac{17}{16}\right)^2-1\right\}=\pi\times 30\times\frac{33}{256}\text{ cub. ft.,}$$

$$=\frac{22}{7}\times 30\times\frac{33}{256}=12\tfrac{69}{448}\text{ cub. ft.}$$

By the second method, the radius of the cylinder which is a mean between the outer and inner radii, $=1\frac{1}{32}$ ft.; and therefore the *surface* of that cylinder

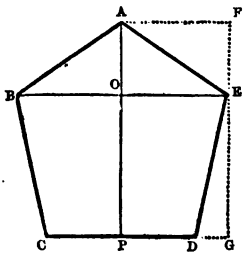
$$=2\pi\times 30\times\frac{33}{32}\text{ sq. ft.}$$

Also the thickness of the metal $=\frac{3}{4}$ in. $=\frac{1}{16}$ ft.;

$$\therefore\ \textit{vol.}\text{ of iron}=\pi\times 30\times\frac{33}{16}\times\frac{1}{16}\text{ cub. ft.,}$$

$$=\pi\times 30\times\frac{33}{256}\text{ cub. ft., as before.}$$

298. *To find the content, or volume, of a haystack, (1) with circular base, (2) with oblong base.*



I. Let $ABCDE$ be a vertical section through the middle of a stack whose base is circular, and the sides diverging upwards in the usual way; the upper part will be a *cone*, whose height is AO, and the *radius* of its base BO; the lower part will be a *frustum* of a *cone* inverted, whose height is OP. Let FEG be a vertical line through E, meeting the horizontals AF, and CD produced, in F and G. Let AO, or FE, be measured, and also EG. Since the radii OE, PD, cannot be directly measured, measure the *circumferences* of the circular sections through E and D; then if c be the circumference of the *mid-section* parallel to the base, and C_1, C_2, those through E and D,

$$c=\frac{C_1+C_2}{2}, \text{ and its radius}=\frac{c}{2\pi};$$

$$\therefore \text{ the area of the } \textit{mid-section} = \pi\times\frac{c^2}{4\pi^2}=\frac{c^2}{4\pi},$$

and the *vol.* of the lower portion of the stack $=\frac{c^2}{4\pi}\times EG$.

Also, the *volume* of the upper, or conical, portion of the stack

$$=\frac{1}{3}\pi\times EF\times OE^2, \text{ where } OE=\frac{C_1}{2\pi},$$

$$=\frac{EF\times C_1^2}{12\pi};$$

$$\therefore \text{ whole } \textit{vol.} \text{ of stack} = \frac{c^2}{4\pi}\times EG+\frac{C_1^2}{12\pi}\times EF.$$

II. Let the base and section be *oblong*, instead of circular, and the upper part still terminate in a point; then we have a *pyramidal*, instead of a *conical*, upper portion; and its *volume* $=\frac{1}{3}$ *height* $\times$ *area of section at its base.*

Also, the lower portion is a *frustum* of a *pyramid*, instead of a *cone;* and its *volume*

$=$ ht. $\times$ area of mid-section,

$=\frac{1}{2}$ *ht.* $\times$ *sum of areas of its highest and lowest sections.*

III. If the base be as in the last case, and the upper portion do not terminate in a *point*, but in a *ridge* parallel and equal to the length of the stack, then the upper part will be a triangular *prism*, of which ABE is the base, and its length the length of the ridge.

Hence, *volume of prism* $=$ length $\times$ area of ABE,

$=$ length of ridge $\times AO\times BO$.

The lower part is a pyramidal *frustum*, as in the last case.

Ex. 1. The heights of the upper and lower portions of a round stack are $4\frac{1}{2}$ ft. and 7 ft. respectively; and the girths of the highest and lowest parts of the frustum are 30 ft. and 24 ft.; find the *volume* of the stack.

The circumference of the mid-section of the frustum $=\dfrac{30+24}{2}=27$;

$$\therefore \textit{vol.}\text{ of frustum}=\frac{7\times(27)^2}{4\pi},$$

$$=\frac{7\times 729}{4\times\frac{22}{7}}=\frac{35721}{88}\text{ cub. ft.},$$

$$=406\text{ cub. ft., } \textit{nearly}.$$

$$\text{And } \textit{vol.}\text{ of cone}=\frac{4\frac{1}{2}\times 900}{12\pi}=\frac{4\frac{1}{2}\times 75\times 7}{22},$$

$$=\frac{2382\cdot 5}{22}\text{ cub. ft.},$$

$$=108\text{ cub. ft. } \textit{nearly};$$

$$\therefore \text{ the whole } \textit{volume}=514\text{ cub. ft., } \textit{nearly}.$$

$$=19\tfrac{1}{27}\ \textit{cub. yds.}\ \ldots\ldots$$

Ex. 2. A stack stands on a rectangular base 13 ft. in length, and 9 ft. in width; the horizontal section at the eaves is 16 ft. in length, and 11 ft. in width; and the height of this portion is 7 ft. The upper portion terminates in a *point*, and its height is $4\frac{1}{2}$ ft. Find the volume of the stack.

Area of highest section of the *frustum* $=16\times 11=176$ sq. ft.

.... lowest .... $=13\times\ 9=117$ ......

$$\text{Area of mid-section}=14\tfrac{1}{2}\times 10=145\text{ sq. ft.};$$

$$\therefore \textit{vol.}\text{ of lower portion}=7\times 145=1015\text{ cub. ft.}$$

$$\text{Also } \textit{vol.}\text{ of upper pyramidal portion}=\frac{4\frac{1}{2}\times 16\times 11}{3},$$

$$=264\text{ cub. ft.}$$

$$\therefore \text{ whole } \textit{volume}=1279\text{ cub. ft.}$$

$$=47\tfrac{10}{27}\ \textit{cub. yds.}$$

Ex. 3. The same data as in Ex. 2, except that the upper portion, instead of tapering to a *point*, terminates in a ridge of the same length as the horizontal section which forms its base.

Vol. of lower portion (as in Ex. 2) = 1015 cub. ft.

.... upper ....(being a prism) $= \dfrac{16 \times 11 \times 4\frac{1}{2}}{2}$ cub. ft.

$= 396$ cub. ft.;

$\therefore$ whole *volume* $= 1411$ cub. ft. $= 52\frac{7}{27}$ *cub. yds.*

NOTE. A cubic foot of *old* hay will weigh about $8\frac{1}{2}$ lbs., on the average, as proved by experiment. Hence the *weight* of a stack will readily be found, when its *volume*, or *content*, has been determined.

Two cwt. per cubic yard will not be far wrong.

EXERCISES N.

(1) Find the cost of a block of stone in the form of a parallelopiped, whose edges are 3, 5, and 8, feet, at 2*s.* 2*d.* per cubic foot. Ans. £13.

(2) What length must I cut off from a plank 2 ft. broad, and $1\frac{1}{2}$ ft. thick, for the sum of £2. 5*s.*, at the rate of 10*d.* per cubic foot? Ans. 18 ft.

(3) What would the painting, of the whole piece cut off in the last Ex., cost, at 1*d.* per square foot?

Ans. 11*s.*

(4) The bottom of a cistern contains 7 sq. ft. 101 sq. in.; how deep must it be to hold 82 gallons, if $277\frac{1}{4}$ cub. in. make 1 gallon? Ans. 1 ft. $8\frac{1}{2}$ in.

(5) A right-angled triangle, whose sides are 3, 4, and 5, inches, is made to turn round upon the side whose length is 4 in., thus describing a *right cone;* find the surface and volume of the cone.

(1) Ans. $47\frac{1}{7}$ sq. in. (2) Ans. $37\frac{5}{7}$ cub. in.

(6) A rectangular parallelogram, 7 inches long, and 1 inch broad, is turned round about one of its longer

sides, and describes a *cylinder;* find the surface and volume of the cylinder.

(1) Ans. 44 sq. in. (2) Ans. 22 cub. in.

(7) A cylindrical shaft, 105 yds. deep, and 2 yds. wide, was to be excavated, at the rate of £1. per yard in depth; but the rate is afterwards changed to one of 6*s*. 8*d*. per cubic yard excavated; what difference is there in the cost? Ans. It costs £5. more.

(8) A right *prism*, whose ends are equilateral triangles, having their sides each $3\frac{1}{2}$ in., is 16 in. long; find its surface and volume.

(1) Ans. 1 sq. ft. 34·6 sq. in. nearly.
(2) Ans. 84·868 cub. in.

(9) An *oblique prism* has a polygonal base of the form described in (226), where the diagonal $AD=4{\cdot}5$ in., and $AC=4{\cdot}8$ in.; also the perpendiculars Bb, Dd, Ee, are 1·2, 2·5, and 1·6, inches, respectively; and the perpendicular height of the prism is 10 in.; find its volume.

Ans. $124\frac{4}{5}$ cub. in.

(10) A *pyramid* of marble has for its base a regular hexagon, whose side is 1 ft.; and the height of the pyramid is 9 ft.; what is the cost, at 10*s*. per cubic foot?

Ans. £3. 17*s*. 11·28*d*.

(11) Some blocks of wood, 1 foot high, and having their ends 4 inches square, are cut into hexagonal *prisms*, with as little waste as possible; find the cost per 1000, at the rate of 2*s*. 6*d*. per cubic foot of manufactured material. Ans. £9. 0*s*. 5*d*.

(12) An hour-glass is made of two equal cones joined at their vertices; the vertical angle is 60°, and the depth of the sand when level in one of the cones is 3 inches; find the volume of sand which must pass into the lower cone per minute, so that the upper cone may be emptied in 1 hour. Ans. $\frac{11}{70}$ cub. in.

(13) Find the cost of lining a cylindrical shaft, 30 feet deep, and $1\frac{3}{4}$ yards broad, with wood 3 inches thick, supposing the cost of material and labour to be at the rate of 1*s*. 9*d*. per cubic foot. Ans. £10. 6*s*. 3*d*.

(14) A cubical mass of metal, whose edge is 3·35 inches, is drawn out into a cylindrical wire 67 inches long; find the area of a section of it perpendicular to its length. Ans. ·561125 sq. in.

(15) The adjacent edges of a rectangular box are $3{\cdot}\dot{4}2857\dot{1}$, $5{\cdot}\dot{1}4285\dot{7}$, and $10{\cdot}\dot{2}8571\dot{4}$, inches; find the cost of gilding its exterior at $1\frac{3}{4}d$. per square inch.

Ans. £1. 10*s*. $10\frac{2}{7}d$.

(16) A solid spherical ball of copper, one foot in diameter, is hammered into a circular plate of one inch uniform thickness. Find the diameter of the plate.

Ans. 2·828 feet.

(17) How many bullets of a quarter of an inch in diameter can be cast from the metal of a spherical ball 3 inches in diameter, supposing no waste in the process? Ans. 1728.

(18) A river with an average depth of 30 feet, and 200 yards wide, is flowing at the average rate of 4 miles an hour; find how many cubic feet of water run into the sea per minute; also the number of tons, supposing a cubic foot of water to weigh 1000 ounces.

(1) Ans. 6,336,000 cub. ft.
(2) Ans. $176785\frac{5}{7}$ tons.

(19) What is the number of cubic feet in the volume of an hexagonal room, each side of which is 20 ft. long, and the walls 30 ft. high, and which is finished above with a roof in the form of an hexagonal pyramid 15 ft. high? Ans. 36372 cub. ft.

(20) Find the cost of painting the walls and ceiling of the room, described in the last Ex., at 1*s*. per sq. yd.

Ans. £27. 12*s*. 9*d*.

(21) What is the solid content of a sphere, whose lineal circumference is $6\frac{2}{7}$ yds.? Ans. 4 cub. yds. $5\frac{1}{7}$ ft.

(22) What is the solid content of a sphere, when its surface is equal to that of a circle 8 yds. in diameter?

Ans. 33 cub. yds. $14\frac{1}{7}$ ft.

(23) Required the cost of a globe of 25 in. diameter, which is to be paid for at 6*d*. the square inch on the surface. Ans. £49. 2*s*. $1\frac{5}{7}d$.

(24) A haystack, $11\frac{1}{2}$ ft. high, has an oblong base 20 ft. long, and 8 ft. broad; the sides of the rectangular horizontal section 9 ft. from the ground through the eaves are 22 ft. and 8·8 ft.; the part above the eaves forms a triangular prism 22 ft. long; find the whole weight of the stack, if 200 cubic feet of the hay weigh 1 ton. Ans. 9·154 tons.

(25) A cylindrical basin 25 ft. in diameter, 4 ft. deep, and $\frac{5}{8}$ths filled with water, is drained by means of a 3 in. pipe, through which the water flows at an average rate of 2 miles per hour; shew that it will continue flowing for 2 hours $22\frac{1}{22}$ minutes.

MISCELLANEOUS EXERCISES.

(1) Find the area of a floor 31 ft. 9 in. long, and 18 ft. 7 in. broad. Ans. 590 sq. ft. 3 sq. in.

(2) A square floor, whose side is 15 yds., is covered by 7200 equal square tiles; what is the length of a side of each tile? Ans. 6·363 inches.

(3) A chess-board having 8 squares along each side is 18 inches square. Find the length of a side of one of its squares. Ans. $2\frac{1}{4}$ inches.

(4) One hundred thousand men are drawn up in a square: how much space will they occupy, if to each man is allowed 2 ft. 3 in., by 1 ft. 9 in.*?

Ans. 43750 sq. yds.

(5) If the men in the last Ex. were drawn up in an *oblong* whose sides are in the proportion of 10 to 1, each man covering 2 ft. square, what would be the *periphery* of the *oblong?*

Ans. $\frac{5}{6}$ths of a mile.

(6) Shew, without assuming any *Rule*, that the area of the rectangle, whose adjacent sides are $7\frac{1}{12}$ ft., and $5\frac{3}{4}$ ft., is equal to $40\frac{35}{48}$ sq. ft.

* To avoid the frequent repetition of the words *rectangular area* it is usual, as here, simply to insert *by* between the numbers expressing the *length* and *breadth* of such an area.

(7) Shew, by a *diagram*, that $30\frac{1}{4}$ sq. yds. = 1 sq. pole.

(8) Prove, by *diagrams*, that $(\frac{3}{4}\text{ ft.})^2 = \frac{9}{16}$ sq. ft.; and that $\frac{1}{8}$ ft. $\times\ \frac{1}{5}$ ft. $= \frac{1}{40}$ sq. ft.

(9) Determine, by a *diagram*, how many equal squares, of $1\frac{1}{8}$ inches side, can be obtained from a rectangle 72 inches long, and 45 inches broad. Ans. 2560.

(10) A lawn is 70 yds. by 32 yds.; find the cost of laying it down with pieces of turf, each 15 in. by 9 in., at 10*s*. the gross. Ans. £74. 13*s*. 4*d*.

(11) A roof 45 ft. by 27 ft. is covered with slates, each 18 in. by 9 in. How many will be required?

Ans. 1080.

(12) The cost of paving a floor with flags, each $18\frac{1}{2}$ in. by $15\frac{1}{2}$ in., at 7*d*. per square foot, comes to £33. 9*s*. 1*d*.; how many flags were there in the floor?

Ans. 576.

(13) A field of $7\frac{1}{2}$ acres is planted in rows at uniform distances of 15 inches; find the number of plants required for the whole field, if in each row the plants are half a yard apart. Ans. 174240.

(14) A field $40\frac{1}{2}$ poles by 24 poles is divided into 72 equal plots; find the number of square yards in each plot, and express the result as the decimal part of an acre. (1) Ans. $408\frac{3}{8}$. (2) Ans. ·084375.

(15) The walls of a room $8\frac{1}{2}$ yds. by 5 yds., and 11 ft. high, are painted at 9*d*. a square yard; what is the whole cost? Ans. £3. 14*s*. 3*d*.

(16) A straight road, 45 ft. wide, and a furlong in length, is cut off the side of a field of 4 acres; how much is there left for cultivation?

Ans. 3 ac. 1r. $10\frac{10}{11}$ p.

(17) What length must be cut off from a plank $9\frac{1}{8}$ inches wide, to make a door whose face is 16 sq. ft.?

Ans. $21\frac{3}{73}$ ft.

(18) If the side of a square be $8\frac{1}{4}$ ft., what decimal part of a rood is its area? Ans. ·00625.

(19) The area of a square picture is $2\frac{1}{4}$ ft., and the width of the frame is 4 in.; how much wall does it cover? Ans. $4\frac{25}{36}$ sq. ft.

(20) From a square containing 1 acre there are subtracted 32 rectangular plots, each 12·6 yds. long, and 10·5 yds. broad; how much is left? Ans. 606·4 sq. yds.

(21) Twenty shutters 9 ft. high, are to be made to cover a shop window, the area of which is 40 sq. yds. 3 ft. What must be the breadth of each shutter?

Ans. $2{\cdot}01\dot{6}$ ft.

(22) Find the areas of the triangles whereof the sides are as follows:

(1)	18, 15, 20;	(1) Ans. 129·75.
(2)	36, 48, 54;	(2) Ans. 846·9.
(3)	15·5, 30, 27·7;	(3) Ans. 212·98.
(4)	6·25, 3·9, 4·17;	(4) Ans. 7·9693.

(23) Given the following lengths of the sides and perpendiculars upon them from the centres of certain polygons; find their areas.

	No. of sides.	Length of side. ft.	Perp. ft.	Area. sq. ft.
(1)	5	3·25	2·45	(1) Ans. 19·90625.
(2)	6	15	13	(2) Ans. 585.
(3)	7	21·5	22·6	(3) Ans. 1700·65.
(4)	10	17·5	21	(4) Ans. 183·75.
(5)	10	10	15·3	(5) Ans. 765.
(6)	12	1·75	3·73	(6) Ans. 39·165.

(24) Find the areas of the irregular polygons, of which the sides and diagonals are as follows.

[NOTE.—The diagonals are all drawn from the extremity of that side whose measure stands first.]

	No. of sides.	Length of sides.	Length of diagonals.
(1)	4	5, 4·5, 5·5, 7;	8,
(2)	5	5, 4·5, 3·4, 2·8, 7;	8, 9.
(3)	6	3, 3·5, 3·6, 6, 2·8, 5;	6, 8, 6·8.

(1) Ans. 29·094. (2) Ans. 31·55. (3) Ans. 39·89.

(25) A tenant has £78. 2*s*. 6*d*. allowed him for draining a rectangular field by a channel traversing the diagonal; how much per lineal yard may he expend upon the drain without loss, if the sides of the field be 100 yards, and 75 yards? Ans. 12*s*. 6*d*.

(26) Find the relation between the sides of a right-angled triangle, whereof one of the acute angles measures 30°. Ans. $1 : \sqrt{3} : 2$.

(27) The sides of a triangle are equimultiples of 3, 4, and 5; shew that its area is 6 times the *square* of the multiple.

(28) Shew that the ratio of the side of a square to its diagonal is 29 : 41 *nearly*.

(29) A railway platform has two of its opposite sides parallel, and its other two sides equal; the parallel sides are 80 ft. and 92 ft. respectively; the equal sides are 10 ft. each; what is its area? Ans. 688 sq. ft.

(30) A field is bounded by four straight lines, of which two are parallel; if the sum of the parallel sides be 625 links, and their perpendicular distance be 160 links, what is the content of the field? Ans. $\frac{1}{2}$ acre.

(31) A rectangular garden is to be cut from a rectangular field, so as to contain a quarter of an acre. One side of the field is taken for one side of the plot, and measures exactly 3·5 chains; how long must the other side be? Ans. Five-sevenths of a chain.

(32) The side of a rhombus is 10 ft., and the longer diagonal is 16 ft.; find the other diagonal, and the area of the rhombus. (1) Ans. 12 ft.; (2) Ans. 96 sq. ft.

(33) Upon the base of an equilateral triangle, whose side is 6 ft., another triangle is described, one-third of the original triangle in area, find its perpendicular height.

Ans. 1·732 ft.

(34) An equilateral triangle has a perimeter of 375 links; find its area as a decimal part of an acre.

Ans. ·06765625.

(35) Find the cost of covering with asphalte, at 8*d.* per sq. yd., a triangular plot, whose sides are 40, 36, and 27·5 yds. Ans. £16. 1*s.* 3*d.*

(36) Find the area of the largest square which can be cut out of a circle whose radius is 1 foot.

Ans. 2 sq. ft.

(37) The largest possible circle is cut out of an area of 15 ft. square; find the area of each of the corners remaining. Ans. $12\frac{3}{56}$ sq. ft.

(38) Two equal circles touch each other, and a cord tightly encloses them both without crossing itself; find the length of the cord, and the area enclosed by it, in terms of the radius.

(1) Ans. $10\frac{2}{7}$ rad. (2) Ans. $7\frac{1}{7}\times(\text{rad.})^2$.

(39) Two equal circles, of 1 inch radius, are distant 2 inches from each other, and a cord passes tightly round them, crossing between them, and in contact with two-thirds of each circumference; find the length of the cord, and the area enclosed by it.

(1) Ans. 15·308 in. (2) Ans. 7·654 sq. in.

(40) Find how many circles of $\frac{1}{2}$ in. radius could be made from another of 1 foot radius, supposing the whole area of the larger circle could be used up. Ans. 576.

(41) If a pound's worth of silver, in sixpences, reaches 25 inches, when the coins are placed side by side in a straight line, what is the diameter of each coin, and the total surface covered by them?

(1) Ans. $\frac{5}{8}$ in. (2) Ans. $12\frac{31}{112}$ sq. in.

(42) The largest possible square is cut out of a given quadrant; compare the area of the square with that of the remainder of the quadrant. Ans. 7 : 4.

(43) The corner of the leaf of a book is turned down twice, so that the lines of folding are parallel, and form, with the edges of the book, two similar right-angled triangles, whose heights are as 1 to 2; if the base and height of the smaller triangle are 2·5 in., and 1·75 in., respectively, find the area of the larger triangle.

Ans. $8\frac{3}{4}$ sq. in.

(44) The external circumference of a flat ring is 9 ft. 2 in., and its width 1 inch; find the internal diameter, and the area of the ring.

(1) Ans. 2 ft. 9 in. (2) Ans. $106\frac{6}{7}$ sq. ft.

(45) A kerb, 9 in. broad, is put to a well 7 ft. in diameter, and costs 7*s.* 9*d.* At how much is that per square foot, reckoning only the upper surface?

Ans. $5\frac{1}{11}d.$

(46) If the area of a sector be 10 sq. ft., and the radius 5 ft., what is the number of degrees in the angle at the centre? Ans. $45^\circ\ 50'\ 11{\cdot}8''$.

(47) Find the area of a sector of a circle, when the diameter is 7·2 ft., and the arc of the segment subtends an angle of $10{\cdot}5^\circ$ at the centre. Ans. 1·188 sq. ft.

(48) A circle rolls on the circumference of another, so that the circumference of the smaller one always passes through the centre of the larger; compare their areas.

Ans. 1 : 4.

(49) The paving of a semicircular alcove at 2*s.* 6*d.* a foot comes to £5; what was the length of the semicircular arc? Ans. 15·85 ft.

(50) If the minute-hand of a clock, 3 ft. long, passes over an arc of $1\frac{4}{7}$ ft. in 5 minutes, what must be the length of the hour-hand of the same dial, if it passes over an arc of equal length in $1\frac{1}{2}$ hrs.? Ans. 2 ft.

(51) In the last Example what will be the areas swept out by each of the hands in 25 minutes?

(1) Ans. $\frac{55}{126}$ sq. ft. (2) Ans. $11\frac{11}{14}$ sq. ft.

(52) Two sides of a triangle are 17·6 yds. and 8·5 yds., and include an angle of 30°; find the area of the triangle. Ans. 37·4 sq. yds.

(53) The area of a circle is 154 sq. ft.; find the length of the side of the inscribed equilateral triangle.

Ans. $7\sqrt{3}$ ft.

(54) Find the base and perpendicular height of a triangle whose area shall be nearly equal to that of a circle of radius $1\frac{3}{4}$ ft. Ans. Base $= 5\frac{1}{2}$ ft.

Height $= 3\frac{1}{2}$ ft.

(55) Supposing the radius of the earth, as seen from the Moon, subtends an angle of $57\frac{3}{11}'$; find the distance of the Earth from the Moon, the Earth's diameter being taken as 8000 miles. Ans. 240,000 miles.

(56) Given a circle; how would you find a circle which should have exactly half the area of the given one?

(57) If a pressure of 15 lbs. be applied to every square inch of a circular plate 3 feet in diameter; what is the total pressure? Ans. 6 tons 16 cwt. $42\frac{2}{7}$ lbs.

(58) Two circular plates, each an inch thick, the diameters of which are 6 in. and 8 in. respectively, are melted, and formed into a single circular plate an inch thick; find its diameter. Ans. 10 inches.

(59) The diameter of a circular saw is 34 inches; what length of slit must be made in the bench, that the highest point of the saw may stand 2 inches above the bench? Ans. Not less than 16 in.

(60) The perimeter of a square is such as to enclose 1296 sq. yds.; how many square yards would a *circle* of the same perimeter include?

Ans. $1649\frac{5}{11}$.

(61) In comparing the lengths of two lines by the method of *Continued Fractions* (p. 249), the successive quotients obtained in the process are 2, 3, 1, 7; find the ratio of the lines as derived from each of these quotients.

Ans. 2, $2\frac{1}{3}$, $2\frac{1}{4}$, $2\frac{8}{31}$.

(62) Find the results corresponding to the quotients 1, 5, 3, 2. Ans. 1, $1\frac{1}{5}$, $1\frac{3}{16}$, $1\frac{7}{37}$.

(63) Suppose the quotients given in Ex. (62) had been obtained in comparing two *circular arcs*, find the successive approximations to the true value of the greater arc, when the smaller one subtends at the centre an angle of 15°. Ans. 15°, 18°, $17\frac{13}{16}^\circ$, $17\frac{31}{37}^\circ$.

(64) Find the results corresponding to the numbers in Ex. (61), when the smaller arc measures $12\frac{1}{2}^\circ$.

Ans. 25°, $29\frac{1}{6}^\circ$, $28\frac{1}{8}^\circ$, $28\frac{7}{31}^\circ$.

(65) What is the length of a *scale*, divided into 10 units, on which the number 9·25 measures 7 inches and 4 tenths? Ans. 8 inches.

(66) If a *scale* be taken of 1 perch to an inch, and the base of the *diagonal scale* be divided into 8 equal parts, and the height into 9 equal parts; what will the smallest subdivision represent?

Ans. $2\frac{3}{4}$ inches.

(67) In a square whose side is 30 inches, a number of equal circles is so placed, that contiguous circles touch each other, and all the outer circles *touch* a side of the square, the diameter of each being *always* an aliquot part of the side of the square, and similar rows of circles are placed throughout the square; shew that whatever be the number of the circles, the portion of the square unoccupied by them will be always the same.

(68) Find the number of degrees in the exterior, and the interior, angles of a regular *decagon*.

(1) Ans. 36°. (2) Ans. 144°.

(69) The French mètre is one ten-millionth part of the fourth of a meridian on the Earth's surface, and is found to be 39·37 inches; find in English miles the length of the quadrant that was measured to obtain the mètre. Ans. 6213·7 miles.

(70) Find the areas of two pieces of land from the following notes; all the dimensions are expressed in links.

298				37	6
176	32		9	29	18
45	15		6	12	11
0			8	0	6

(1) Ans. 8·3584 perches. (2) Ans. 1·1072 perches.

(71) A distance was observed to measure 180 yards on the slope, but only 173·862 on the horizontal; find by means of the Table in (262) the *angle* between the two lines. Ans. 15°.

(72) To what degree of accuracy does a *Vernier* measure, when the unit on the scale is divided into 20

equal parts, and on the vernier 19 of these parts are divided into 20 equal parts. Ans. $\frac{1}{400}$ of the unit.

(73) The cost of 100 bricks whose dimensions are $2\frac{1}{4}$ in. thick, 4 in. broad, and $8\frac{1}{4}$ in. long, is *2s. 3d.*; what will the cost be, reckoned in proportion to the quantity of material, if each dimension be increased by *one-third* of itself? Ans. *5s. 4d.*

(74) Prove that the diagonal and edge of a *cube* are *incommensurable.*

(75) One solid contains $30\frac{3}{8}$ cubic feet, and another $4\frac{1}{2}$ cubic yards; what multiple is the latter of the former?

Ans. 4.

(76) Compare the area of a section of a cube through two opposite edges, with the whole surface of the cube.

Ans. $\sqrt{2} : 6$.

(77) A cylindrical cup is 4 in. deep, and 2 in. in diameter; how often can it be filled from a cylindrical barrel 4 ft. deep, and 30 in. in diameter? Ans. 2700 times.

(78) A cone, whose vertical angle is 120°, and perpendicular height 3 ft., has one third of its height cut off; find the area of the curved surface and ends of the remaining frustum. Ans. 181·38 sq. ft.

(79) A cube has each of its edges diminished by $\frac{1}{100}$th part; compare the *surfaces* of the new cube and the original one. Ans. 97·03 : 100.

(80) If the height of a cubic inch be diminished by $\frac{1}{10}$th; by how much must each side of the square base be increased, so that the whole volume may remain unaltered? Ans. ·054 in.

(81) From the corners of a square piece of cardboard, whose side is 3 in., 4 squares are cut, each 1 in. square, and the remainder is made into a box without lid; what will be its content and outer surface?

(1) Ans. 1 cub. in. (2) Ans. 5 sq. in.

(82) How many gallons of water will a cistern hold, whose interior length is $3\frac{1}{2}$ ft., breadth $2\frac{1}{4}$ ft., and depth 30 in., if $277\frac{1}{4}$ cubic inches make 1 gallon? Ans. 122·7.

(83) If the sides of the cistern in Ex. (82) be constructed throughout one inch in thickness; how many solid feet of material are used in the construction?

Ans. 3 cub. ft. 352 in.

(84) The walls of a cylindrical room 16 ft. high, and 18 ft. in diameter, are painted at $7\frac{1}{2}d.$ per square yard; find the cost. Ans. £3. 2s. $10\frac{2}{7}d.$

(85) A plate of metal, 3 in. square, and ·1 in. thick, is drawn into a wire 100 ft. long; express the measure of the area of a section of the wire, in decimal parts of a square inch. Ans. ·00075 sq. in.

(86) How many cubes, whose edges are $\frac{3}{8}$ in. long, can be contained in a box, whose base is 9 in. by 8 in., and height 15 in.? Ans. 20480.

(87) If gold be beaten out so thin that an ounce of it will form a leaf of 20 square yards, how many of these leaves will make an inch thick, supposing the weight of a cubic foot of gold to be 10 cwt. 95 lbs.?

Ans. 291600.

(88) A gold wire of ·01 of an inch in thickness is bent into a circular ring one inch in diameter; if the area inclosed by the ring be gilded with a weight of gold equal to the weight of the ring, what will be the thickness of the gilding? Ans. $\cdot 0001\frac{1}{7}$ in.

(89) Shew that the volume of a sphere, whose radius is 6 in., is equal to the sum of the volumes of the spheres whose radii are 3, 4, and 5, inches.

(90) The inner and outer circumferences of the base of a hollow cylinder are 3·27 and 3·69 feet; find the area of the *ring* included between them, and the volume of the metal used in constructing the cylinder, if it be 6 ft. high.

(1) Ans. 33·4828 sq. in. (2) Ans. 1 ft. 682·76 cub. in.

(91) Find how much material is wasted in paving a floor 24 ft. by 16 ft., with hexagonal blocks 9 inches long, of which the hexagon has each side 2 inches long, and which are cut out of cylindrical blocks 4 inches thick, and 9 inches long. Ans. 39·168 cub. ft.

(92) A cylindrical boiler, 16 ft. long, and 2 ft. in diameter, with hemispherical ends, in addition to the above length, has to be covered with felt; what will it cost at $1\frac{1}{2}d.$ per square foot? Ans. $14s.\ 1\frac{5}{7}d.$

(93) It is required to make a cistern, 3·2 ft. long, and 2·6 ft. wide, that shall contain 216 gallons; how deep must the cistern be, if $277\frac{1}{4}$ cubic inches make 1 gallon?

Ans. 4·33...ft.

(94) A rectangular mass of earth is 9·45 yds. long, $3\frac{3}{20}$ yds. broad, and $1\frac{1}{20}$ yds. thick; find the edge of a *cubical* mass of equal volume. Ans. 3·15 yds.

(95) Two cylindrical cups of the same height will hold 9 and 16 pints respectively; what is the content of another of the same height the diameter of whose base is equal to the *sum* of the diameters of the former two?

Ans. 49 pints.

(96) Not only the capacity, but the form also, of the *Imperial Bushel*, is defined by Act of Parliament. Explain the necessity for this enactment.

(97) The Act of Parliament directs, that the *Imperial Bushel* used for *heaped measure* shall be an upright cylinder, the diameter of whose base is not less than twice the height, and that the height of the conical heap shall be at least *three-fourths* of the depth of the bushel, the boundary of its base being the outside of the measure. State fully the effect of not complying with this regulation.

(98) Required the content of a tub, in the form of a frustum of a cone, whose greatest diameter is 60 in., diagonal 66 in., and slant side 30 in.

Ans. 81410 cub. in. nearly,
or $288\frac{1}{2}$ gallons ...

(99) A gentleman wishes to raise his garden 1 foot higher throughout by means of earth dug out of a moat to be formed 8 feet wide round two adjacent sides of it; the garden is 300 feet long and 200 feet broad, and is rectangular. How deep must he dig the moat, supposing it uniform and rectangular? Ans. $14\frac{33}{43}$ ft.

(100) Of what diameter must the bore of a cannon be cast for a ball of 24 lbs. weight, so that it may be one-tenth of an inch more than that of the ball? (See Appendix.) Ans. 5·61 in.

(101) A railway which for some distance has been laid in a straight line at a certain point takes a circular bend for 476 yards, and then proceeds again in a straight line, which deviates from the former by an angle of $12° 38\frac{1}{2}'$; find the radius of the curve.

Ans. 1 mile 397 yards.

(102) In levelling for canals and railways engineers allow a depression of 8 inches per mile for the curvature of the earth. What is the earth's diameter, supposing this to be correct?

Ans. 7920 miles.

(103) A cubic foot of copper is to be drawn into a wire of $\frac{1}{40}$th of an inch diameter; what will be the length of the wire? Ans. $55\frac{1}{2}$ miles *nearly*.

An excellent collection of easy examples, by the Rev. W. N. Griffin, is published by the National Society, and sold at the Depository, Sanctuary, Westminster, at the low price of $1\frac{1}{2}d$. It is exactly adapted to this work, and junior students will find it of great service.

NOTE.

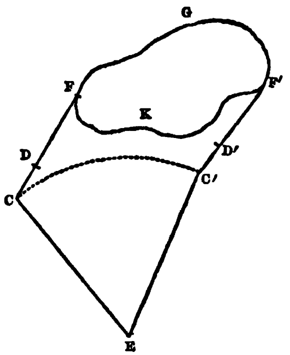
AMSLER'S PLANIMETER. (See p. 314.)

For the following popular explanation of the principle of this instrument, I am indebted to Dr. Alfred Day, of Clifton, near Bristol; though I have not adhered precisely to that gentleman's own words:—

Suppose a rod of wood, or brass, carried a wheel attached to it revolving at right angles to its length, like the rolling *Parallel-Ruler* deprived of one of its wheels, it would, on being moved parallel to itself, describe an area which would obviously be measured, as a rectangle, by the product of the distance moved over by the wheel into the length of the rod, or ruler. And if, in addition to this parallel motion, the ruler were made to rotate, or deviate from its first direction, we could resolve, or separate, the area traced out into two parts, one due to the advance of the ruler parallel to itself, and the other wholly due to rotation, backwards or forwards, that is, diminishing or increasing the previously described area. By means of these *two* motions, and a *third* in the direction of the ruler's length, (which latter will affect neither of the two former, nor add any thing to the area), we can make the end of the ruler trace out any continuous curve or irregular line we please. If then, at the same time, we note the track of the *other* extremity of the ruler, and complete the figure by two straight lines coinciding with the edge of the ruler in its first and last position; and if we see further, that for every partial rotation of the ruler round itself in one part of its course, an equal and opposite partial rotation takes place in the other direction, so that at last the ruler is exactly parallel to its first position, then, in this case, it is plain, that we should at once have a measure of the area above described, viz. *the product of the ruler's length into the rotation of the wheel.*

It will be obvious, that, in practice, the operation here described would be of very limited utility, because in most cases, when we had a given irregular figure to measure, while we made *one* end of the ruler trace out the given boundary, the other end would be tracing a

boundary with which we were in no way concerned, besides the difficulty connected with the two parallel straight lines. The *Planimeter* at once obviates this inconvenience.



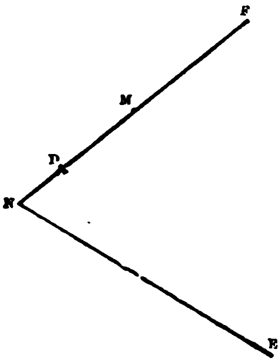
Let $ECDF$ be the instrument; EC the arm *fixed* at E; and jointed at C to the tracer-rod CF; F the tracing point, and D the position of the wheel on CF; $FGF'K$ the area to be measured. Beginning with the tracer-point at F, it is plain that, in passing from F to F', along the boundary FGF', the arm CF will, by means of the three sorts of motion before mentioned, trace out the area $CFGF'C'$, while CE will trace the sector CEC'. Accordingly, the wheel will register all the advance of the arm CF due to parallel motion, together with that which has resulted from rotation round its own axis, when it gets to the extent of its positive progress; certain quantities of rotation, positive and negative, having balanced themselves wholly, and left no record of their existence. Then, as the tracer moves along the boundary $F'KF$, not only will all the sector described by EC' be, as it were, wiped out, but all the area $FKF'C'C$, and all the *rotation* of the tracer-arm round itself is also counter-balanced by a negative rotation, when the tracer has returned to its first position F. In going back, therefore, the wheel will register the actual parallel motion of the tracer-arm, and leave all the rotations balanced.

Hence the area $FGF'K = CFGF'C' - FKF'C'C$,
$=$ *rotation of the wheel* $\times$ *the arm which carries it.*

[Dr Day has further discussed the various details connected with this beautiful instrument, and has proved the truth of its determinations, with admirable skill and

success, in every possible case. But the above will suffice for our purpose here, that is, to give the ordinary student a notion, at least, of the *principle* of the instrument.]

For the *advanced* reader I am permitted to give the following elegant proof of the principle by Professor Adams, of Cambridge, the celebrated astronomer :—



Let E be the fixed point, F the tracer, N the projection of the hinge upon the plane of the paper, D the point in which the plane of the wheel meets NF, M the middle point of NF.

Also let

$$NF=a, \quad EN=b, \quad DM=c.$$

If the boundary of any closed figure be traced out by F, the area of the figure is equal to the algebraical sum of the elementary areas described by the lines EN, NF in passing from any position to a consecutive position, considering an elementary area to be positive when it passes from the left to the right side of the lines, and negative when it passes in the opposite direction.

In any position of the tracer, let ϕ, ψ, be the angles which NF, EN, make with their respective initial positions; and let s be the arc through which the wheel has turned in the same time.

For a consecutive position of the tracer, let ϕ, ψ, and s, become $\phi+\delta\phi$, $\psi+\delta\psi$, and $s+\delta s$, respectively.

Then δs = the resolved part of the motion of the point D perpendicular to the line NF.

Hence the resolved part of the motion of M perpendicular to the same line $=\delta s+c\,\delta\phi$; and therefore the elementary area described by $NF=a(\delta s+c\,\delta\phi)$. Also the elementary area described by $EN=\frac{1}{2}b^2\delta\psi$.

Hence the algebraical sum of the elementary areas described by EN, NF, in passing from their initial positions to any other positions

$$=as+ac\phi+\tfrac{1}{2}b^2\psi.$$

If, when the tracer has passed completely round the boundary of the figure, EN, NF, return to their initial positions without having made a revolution, ϕ and ψ vanish, and the area of the figure $= as$.

If EN, NF, have made a complete revolution, ϕ and ψ become $= 2\pi$, and the area

$$= as + \pi(2ac + b^2).$$

In a given instrument, the area of the rectangle, contained by sides equal to a, and the circumference of the wheel, is known; and as in any case is found by multiplying this area by the number of revolutions, and parts of a revolution, of the wheel.

The quantity to be added to as, when EN, NF, make a complete revolution, is constant, and equal to the area of a circle, the square of the radius of which $= 2ac + b^2$.

$$\text{Since } 2ac + b^2 = (DF + ND)(DF - ND) + EN^2,$$
$$= DF^2 - ND^2 + EN^2,$$

this circle is equal to that which F describes about the centre E, when ED remains perpendicular to NF during the motion; in which case the wheel moves always perpendicularly to its own plane, and does not turn about its axis, so that as vanishes.

In the instrument described in (278), the length a is such, that the rectangle contained by it and a line equal to the circumference of the wheel $= 10$ *square inches*.

When the sliding rod is used, a is increased in the ratio of 1·44 to 1, so that one revolution of the wheel now corresponds to 14·4 square inches, or $\frac{1}{10}$ of a square foot.

APPENDIX.

CERTAIN THINGS TO BE REMEMBERED OR REFERRED TO.

LINEAL MEASURE.

(1)

1 *French*	Metre	= 39·37079	*English* lineal inches.
......	Decimetre	= 3·937079	... ...
......	Centimetre	= 0·3937079	... ...
......	Millimetre	= 0·03937079	... ...
......	Decametre	= 393·7079	... ...
......	Hecatometre	= 3937·079	... ...
......	Kilometre	= 39370·79	... ...
......	Myriometre	= 393707·9	... ...

ANGULAR MEASURE.

(2)

1 *French*	Grade	= 0·9	*English* Degrees.
......	Minute	= 0·009	... ...
......	Second	= 0·00009	... ...

i. e. 100 *Grades* = 90 *Degrees;* 100 French *Minutes* = 1 *Grade;* and 100 French *Seconds* = 1 French *Minute.*

(3) The English *Imperial Gallon* = 277·274 *cubic inches.*

(4) The Imperial *Bushel* = 8 Imperial *Gallons.*

(5) The Imperial Gallon of *Water* weighs 10 lbs. *avoirdupois.*

(6) One *Cubic Inch* of Water weighs 0·036 lbs. *avoirdupois.*

(7) One *Cubic Foot* of Water weighs $62\frac{1}{4}$ lbs. *nearly.*

(8) One *Cubic Inch* of

Gold	weighs	0·7 lbs.	avoirdupois *nearly.*
Silver	...	0·378	... ...
Copper	...	0·321	... ...
Lead	...	0·412	... ...
Wrought Iron	...	0·281	... ...
Cast Iron	...	0·275	... ...
Portland Stone	...	0·092	... ...
Brick	...	0·072	... ...
Common Earth	...	0·049	... ...
Air	...	0·000468	... ...
Water	...	0·036	... ...

NOTE. A *Table of Specific Gravities* is a Table which gives the number of times a given bulk (as a cubic inch, or foot,) of any specified substance weighs the same bulk of water. And, therefore, since a cubic inch of water weighs 0·036 lbs., the weight in lbs. of a cubic inch of any other substance will be readily found by multiplying the number opposite to it in the *Table* by 0·036.

Also, knowing the weight of any body, and its *Specific Gravity* from the *Table*, we can find its *volume*. For having found, as above, the weight of one cubic inch, the whole weight of the body divided by this will obviously give the whole number of cubic inches in it.

TABLE OF SPECIFIC GRAVITIES.

Water	1·000	Ivory	1·826
Air	0·013	Lead	11·352
Brick	2·000	Mercury	13·586
Cork	0·240	Portland Stone	2·570
Copper	8·788	Platinum	20·337
Gold	19·258	Rain Water	0·985
Ice	0·916	Salt-Water	1·026
Iron (cast)	7·207	Silver	10·511
Iron (wrought)	7·788	Zinc	7·100

THE END.

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CORRECTIONS.

PAGE		FOR	READ
221	Ex. (12) Ans. (1)	$1\frac{7}{144}$	19.
...	... Ans. (2)		$\frac{1}{8}$.
...	Ex. (16), all the three Answers should be halved.		
222	Ex. (27) (1) Ans.	6·364	2·1213....
...	Ex. (28) Ans.	1·183	1·1183.
223	Ex. (30) Ans.	8*s*. 3*d*.	16*s*. 6*d*.
298	Line 11	$CB = CD$	$CB = (\sqrt{3} - 1) \times CD$,
	$\therefore PD = CD = CB \div (\sqrt{3} - 1)$, which can be measured.		
301	Ex. (1) Ans.	$6\frac{3}{4}$	$14\frac{90}{121}$.
302	Ex. (18) after 'equal to' insert $\sqrt{\qquad\qquad}$.		
303	Ex. (22) (1) Ans.	$1003\frac{1}{2}$.	$853\frac{1}{2}$.
...	... (2) Ans.	1r. 5·9p.	5·1744p.
320	Ex. (8) (1) Ans.	44	$25\frac{1}{7}$.
...	... (2) Ans.	$49\frac{1}{2}$.	$30\frac{25}{56}$.
343	Ex. (6) (1) Ans. and (2) Ans.	ft.	in.
...	Ex. (9) Ans.	116	124.
...	Ex. (11) Ans.	£10. 8*s*. 4*d*.	£9. 0*s*. 5*d*.
...	Ex. (13)	30 yds.	30 ft.
344	Ex. (16), after 'copper' insert 'a foot in diameter.'		
...	Ex. (19)	155880	36372.
...	Ex. (20)	£26	£27. 12*s*. 9*d*.
345	Ex. (25)	hemispherical	cylindrical.
347	Ex. (21)	1 ft. 6 in.	3 ft.
...	Ex. (22) (4) Ans.	79·693	7·9693.
...	Ex. (23) (4) No. of sides	8	10.
...	Ex. (24) (3) Ans.	45·097	39·89.
348	Ex. (30) Ans.	1 acre	$\frac{1}{2}$ acre.
349	Ex. (43) Ans.	$17\frac{1}{2}$	$8\frac{3}{4}$.
350	Ex. (45)	18*s*. 6*d*.	7*s*. 11*d*.
...	Ex. (46)	5 yds.	5 ft.
...	Ex. (51)		$\frac{55}{126}$.
351	Ex. (61)	2, 5, 1, 7	2, 3, 1, 7.
352	Ex. (65) Ans.	12 in.	8 in.
353	Ex. (79) Ans.	97·03	98·01.
354	Ex. (88) Ans.	·0001$\frac{4}{7}$	·0003$\frac{1}{7}$.
355	Ex. (93) Ans.	4·33	4·165.
...	Ex. (99) Ans.	$14\frac{23}{43}$	$14\frac{97}{127}$.

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